School of Mathematical Sciences
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Examiner: Dr A. Treglown

MTH5105 Differential and Integral Analysis
MID-TERM TEST

Date: 22 Feb 2013 *Time:* 15:10–15:50

Complete the following information:

Name	
Student Number (9 digit code)	

The test has THREE questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish to be marked.

Question	Marks
1	
2	
3	
Total Marks	

Nothing on this page will be marked!

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Question 1.

(a) State the formula for the Taylor polynomial $T_{n,a}$ of degree n of a function f at the point a.

[*10 marks*]

Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = 1/\sqrt{1+2x}$.

(b) Determine the Taylor polynomials $T_{2,0}$ and $T_{3,0}$ of degree 2 and 3, respectively, for f at $a = 0$. [*15 marks*]

(c) Using the Lagrange form of the remainder term, or otherwise, show that

$$T_{3,0}(x) < f(x) < T_{2,0}(x) \quad \text{for all } x > 0 \ .$$

[*10 marks*]

Answer 1.

(a)

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \ .$$

[*10 marks*]

(b) From $f(x) = (1+2x)^{-1/2}$ compute

$$f'(x) = -(1+2x)^{-3/2}, \quad f''(x) = 3(1+2x)^{-5/2}, \quad f'''(x) = -15(1+2x)^{-7/2}.$$

Therefore, $f(0) = 1$, $f'(0) = -1$, $f''(0) = 3$, $f'''(0) = -15$ and [*5 marks*]

$$T_{2,0}(x) = 1 - x + \frac{3x^2}{2} \quad \text{and} \quad T_{3,0}(x) = 1 - x + \frac{3x^2}{2} - \frac{5x^3}{2}.$$

[*5+5 marks*]

(c) Let $x > 0$. Taylor's Theorem tells us that $f(x) = T_{2,0}(x) + R_2$, for some $c \in (0, x)$ where

$$R_2 = \frac{-15(1+2c)^{-7/2}}{3!}x^3 = \frac{-5}{2}\frac{x^3}{(1+2c)^{7/2}}.$$

[*5 marks*]

Since $0 < c < x$ we have,

$$-\frac{5x^3}{2} < R_2 < 0$$

and therefore $T_{3,0}(x) < f(x) < T_{2,0}(x)$, as required. [*5 marks*]

Answer 1. (*Continue*)

Question 2.

(a) Give the definition of $f : \mathcal{D} \to \mathbb{R}$ being differentiable at a point $a \in \mathcal{D}$. [*10 marks*]

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 2x^3 \cos\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Prove that $f'(0) = 0$. [*15 marks*]

(c) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 2x \cos\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Is g differentiable at 0? Briefly justify your answer. [*10 marks*]

Answer 2.

(a) f is differentiable at $a \in \mathcal{D}$ if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. [*10 marks*]

(b) Given any $\varepsilon > 0$, choose $\delta = \sqrt{\varepsilon/2}$. [*5 marks* for choosing a valid δ]

So if $0 < |x| < \delta$ then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{2x^3 \cos(1/x)}{x} \right| = |2x^2 \cos(1/x)| \leq |2x^2| = 2|x|^2 < 2\delta^2 = \varepsilon.$$

[*8 marks*]

Here we have used that $|\cos(1/x)| \leq 1$ for all $x \neq 0$. [*2 marks*]

Hence,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

and so $f'(0) = 0$, as required.

Full marks will also be achieved for the following type of solution:

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{2x^3 \cos(1/x)}{x} \right| = |2x^2 \cos(1/x)| \leq |2x^2| = 2x^2.$$

(Here we have used that $|\cos(1/x)| \leq 1$ for all $x \neq 0$.) $2x^2 \to 0$ as $x \to 0$. Thus, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$ and so $f'(0) = 0$.

(c) g is not differentiable at 0. [*3 marks*]

Indeed, if $x \neq 0$ then

$$\frac{g(x) - g(0)}{x - 0} = \frac{2x \cos(1/x)}{x} = 2\cos(1/x)$$

and $\lim_{x \to 0} 2\cos(1/x)$ does not exist (since given any $a \in [-2, 2]$ there exists an x arbitrarily close to 0 such that $2\cos(1/x) = a$). Thus, $\lim_{x \to 0}(\frac{g(x) - g(0)}{x - 0})$ does not exist and so g is not differentiable at 0. [*7 marks*]

Answer 2. (*Continue*)

Question 3.

(a) State the Mean Value Theorem. [*15 marks*]

(b) Suppose that $0 < a < b$. By applying the Mean Value Theorem to the logarithm function show that

$$1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1.$$

You may assume standard properties of the logarithm function. [*15 marks*]

Answer 3.

(a) MVT: Let f be continuous on $[a, b]$ and differentiable on (a, b).

[*5 marks*]

Then there exists $c \in (a, b)$ such that

[*5 marks*]

$$f'(c) = \frac{f(b) - f(a)}{b - a}\ .$$

[*5 marks*]

(b) Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = \log(x)$. f is continuous on $[a, b]$ and differentiable on (a, b). Thus, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\log b - \log a}{b - a}.$$

[*5 marks*]

Note that $f'(c) = 1/c$ and $\log(b) - \log(a) = \log(b/a)$ therefore

$$\frac{1}{c} = \frac{\log\left(\frac{b}{a}\right)}{b - a}.$$

[*5 marks*]

Since $0 < a < c < b$, we have $1/b < 1/c < 1/a$. Thus,

$$\frac{1}{b} < \frac{\log\left(\frac{b}{a}\right)}{b - a} < \frac{1}{a}.$$

Multiplying by $(b - a)$ gives

$$1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1,$$

as desired. [*5 marks*]

Answer 3. (*Continue*)