MTH5105 Differential and Integral Analysis 2012-2013

Solutions 1

1 Exercises for Feedback

1) Determine at which points the following two functions are differentiable, evaluating the derivative wherever it exists.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto (x+1)|x|$,

   (b) $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto (x-1)|x-1|$.

Solution:

(a) The function f is not differentiable at 0, but is differentiable at all $x \in \mathbb{R} \setminus \{0\}$.

The formula for its derivative is:

$$f'(x) = \begin{cases} 2x+1 & x > 0 \\ \text{undefined} & x = 0 \\ -2x-1 & x < 0\,. \end{cases}$$

To prove non-differentiability at 0, note that

$$\frac{f(x)-f(0)}{x-0} = \frac{(x+1)|x|-0}{x} = \begin{cases} x+1 & x > 0 \\ -x-1 & x < 0 \end{cases}$$

so that the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x}$ does not exist
(because $\lim_{x\searrow 0} \frac{f(x)-f(0)}{x} = 1 \neq -1 = \lim_{x\nearrow 0} \frac{f(x)-f(0)}{x}$).
To prove differentiability of f at $x \in \mathbb{R} \setminus \{0\}$, one method is to use the definition

$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

to write

$$f(x) = \begin{cases} (x+1)x & x > 0 \\ -(x+1)x & x < 0 \end{cases},$$

and then use that the function $x \mapsto (x+1)x$ is differentiable, with derivative equal to $2x+1$, to deduce that f is differentiable everywhere on $\mathbb{R} \setminus \{0\}$, with

$$f'(x) = \begin{cases} 2x+1 & x > 0 \\ -2x-1 & x < 0\,. \end{cases}$$

An alternative proof is to work directly with the definition of the derivative, as follows:

For $a > 0$, we have

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(x+1)|x|-(a+1)|a|}{x-a} \\
&= \lim_{x\to a} \frac{(x+1)x-(a+1)a}{x-a} = \lim_{x\to a}(x+a+1) = 2a+1\ .
\end{aligned}$$

(Strictly speaking, some argument is needed as to why we can replace $|x|$ by x when calculating the above limit. It suffices to say that x becomes positive as $x \to a$ when $a > 0$. Or, more formally, in the definition of the limit one can replace δ by $\delta' = \min\{\delta, a\}$ as then $|x-a| < \delta'$ implies $|x-a| < a$ and thus $x > 0$. However, I don't require this degree of formality.)

Similarly, for $a < 0$ we have

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(x+1)|x|-(a+1)|a|}{x-a} \\
&= \lim_{x\to a} \frac{(x+1)(-x)-(a+1)(-a)}{x-a} = \lim_{x\to a}(-x-a-1) = -2a-1\ .
\end{aligned}$$

(Again, some argument is needed as to why we can replace $|x|$ by $-x$ when calculating the limit; it suffices to say that x becomes negative as $x \to a$ when $a < 0$).

(b) The function g is differentiable at all points of $\mathbb{R}$, and its derivative is given by the formula

$$g'(x) = \begin{cases} 2(x-1) & x > 1 \\ 0 & x = 1 \\ -2(x-1) & x < 1 \end{cases}$$

or, in other words, $g'(x) = 2|x-1|$.

How do we prove this?

It is relatively easy to see that g is differentiable at points $x \neq 1$: for $x > 1$ we note that $g(x) = (x-1)^2$, which is differentiable with derivative $2(x-1)$, while for $x < 1$ we note that $g(x) = -(x-1)^2$, which is differentiable with derivative $-2(x-1)$.

A more careful proof of this, working directly with the definition of derivative, is as follows:

For $a > 1$, we have

$$\begin{aligned}
g'(a) &= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{(x-1)|x-1|-(a-1)|a-1|}{x-a} \\
&= \lim_{x\to a} \frac{(x-1)^2-(a-1)^2}{x-a} = \lim_{x\to a}(x+a-2) = 2(a-1)\ .
\end{aligned}$$

while for $a < 1$, we have

$$\begin{aligned}
g'(a) &= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{(x-1)|x-1|-(a-1)|a-1|}{x-a} \\
&= \lim_{x\to a} \frac{-(x-1)^2+(a-1)^2}{x-a} = \lim_{x\to a}(-x-a-2) = -2(a-1)\ .
\end{aligned}$$

Lastly, to prove differentiability at $a = 1$, we have

$$g'(1) = \lim_{x\to 1} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1} \frac{(x-1)|x-1|-0}{x-1} = \lim_{x\to 1} |x-1| = 0\ .$$

2 Extra Exercises

2) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at zero and find $f'(0)$.

Solution:

Consider the difference quotient

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin(1/x^2) - 0}{x} = x \sin(1/x^2) \, .$$

Given any $\varepsilon > 0$, let $\delta = \varepsilon$. Then given any $0 < |x| < \delta$,

$$|x \sin(1/x^2) - 0| = |x| |\sin(1/x^2)| \leq |x| < \delta = \varepsilon.$$

(here we used that $|\sin(1/x^2)| \leq 1$ for all $x \neq 0$). So by definition,

$$\lim_{x \to 0} x \sin(1/x^2) = 0.$$

Therefore f is differentiable at zero with $f'(0) = 0$.

3) Let $f : [-1, 1] \to \mathbb{R}$ be continuous on $[-1, 1]$, differentiable at zero and $f(0) = 0$. Show that the function

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

is continuous at zero.

Is g continuous for $x \neq 0$?

Deduce that there is some number M such that

$$f(x)/x \leq M \quad \text{for all} \quad x \in [-1, 1] \setminus \{0\} \, .$$

Solution:

A function g is continuous at a if $\lim_{x \to a} g(x) = g(a)$.

With $a = 0$, this gives

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = g(0)$$

so g is continuous at 0.

For $x \neq 0$, g is continuous since it is a quotient of continuous functions.

By the boundedness principle (recall this from MTH5104: Convergence and Continuity!), a continuous function on a closed interval attains it maximum and minimum.

Therefore there exists a number M such that $g(x) \leq M$ for all $x \in [-1, 1]$.

4) Give an example of a function that is differentiable on (a, b) but that cannot be made differentiable on $[a, b]$ by any definition of $f(a)$ or $f(b)$. Can you give an example where f is bounded?

Solution:

There are many possible examples, for example if we define $f(x) = \dfrac{1}{(x-a)(b-x)}$ on (a, b) then f is differentiable on (a, b) but cannot be made to be continuous at a or b by any definition of $f(a)$ or $f(b)$.

We get a bounded function if we compose this with sin, i.e. if we define

$$f(x) = \sin\left(\frac{1}{(x-a)(b-x)}\right)$$

on (a, b), then again f is differentiable on (a, b) but cannot be made to be continuous at a or b by any definition of $f(a)$ or $f(b)$. However, once $f(a)$ and $f(b)$ are defined, the resulting function is clearly bounded on $[a, b]$.