MTH5105 Differential and Integral Analysis 2012-2013

Exercises 2

There are two sections. Questions in Section 1 will be used for feedback. Questions in Section 2 are voluntary but highly recommended.

1 Exercises for Feedback

1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Show that

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is constant. *Hint: try to compute the derivative of f first.*

2 Extra Exercises

3) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with

$$f' = g \qquad \text{and} \qquad g' = -f \ .$$

Show that between every two zeros of f there is a zero of g and between every two zeros of g there is a zero of f.

4) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable ($f'' = (f')'$) with

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad f(1) = 1 \ .$$

Show that there exists $c \in (0, 1)$ such that $f''(c) > 1$.

5) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Prove that f is decreasing on $[a, b]$ (i.e. $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$) if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.

6) Suppose that f is continuous on $[0, 1]$, differentiable on $(0, 1)$, and $f(0) = 0$. Prove that if f' is decreasing on $(0, 1)$, then the function $g : (0, 1) \to \mathbb{R}$ given by $g(x) = f(x)/x$ is decreasing on $(0, 1)$.

The deadline is 11am on Tuesday 29th January. Please hand in your coursework to the orange coursework box on the second floor. Coursework will be returned during the exercise class immediately following the deadline.