MTH5105 Differential and Integral Analysis 2012-2013

Solutions 2

1 Exercises for Feedback

1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Show that

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

Solution:

Let $x, y \in \mathbb{R}$. The required inequality is clearly true if $x = y$ (in which case it becomes an equality), so without loss of generality suppose $x < y$. Then f is continuous on $[x, y]$ and differentiable on (x, y), so we can apply the Mean Value Theorem to f on the interval $[x, y]$. The MVT implies that there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) .$$

But $|f'(c)| \leq 1$ by hypothesis, therefore

$$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq 1 ,$$

which implies $|f(y) - f(x)| \leq |y - x|$, as required.

2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is constant. *Hint: try to compute the derivative of f first.*

Solution:

From the inequality it follows that for all $x, a \in \mathbb{R}$ (with $x \neq a$),

$$\left| \frac{f(x) - f(a)}{x - a} \right| \leq |x - a| .$$

We claim this implies f is differentiable at a, with $f'(a) = 0$. To see this, just note that for any $\varepsilon > 0$ we may choose $\delta = \varepsilon$, and then if $0 < |x - a| < \delta$ then

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| = \left| \frac{f(x) - f(a)}{x - a} \right| \leq |x - a| < \delta = \varepsilon .$$

In other words,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0 .$$

By Theorem 2.5 it follows that f is constant when restricted to any closed interval; hence $f : \mathbb{R} \to \mathbb{R}$ is constant.

2 Extra Exercises

3) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with

$$f' = g \qquad \text{and} \qquad g' = -f \ .$$

Show that between every two zeros of f there is a zero of g and between every two zeros of g there is a zero of f.

Solution:

Choose $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = f(b) = 0$.
As f is differentiable on $\mathbb{R}$, the assumptions of Rolle's Theorem are satisfied on $[a, b]$, i.e. f continuous on $[a, b]$ and differentiable on (a, b).

Therefore there exists $c \in (a, b)$ such that $f'(c) = 0$.

As $f' = g$, $g(c) = f'(c) = 0$.

An analogous argument is valid with f and g exchanged.

4) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable ($f'' = (f')'$) with

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad f(1) = 1 \ .$$

Show that there exists $c \in (0, 1)$ such that $f''(c) > 1$.

Solution:

As f is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, 1]$, i.e. f continuous on $[0, 1]$ and differentiable on $(0, 1)$.

Therefore there exists $d \in (0, 1)$ such that

$$f'(d) = \frac{f(1) - f(0)}{1 - 0} = 1 \ .$$

As f' is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, d]$, i.e. f' continuous on $[0, d]$ and differentiable on $(0, d)$.

Therefore there exists $c \in (0, d)$ such that

$$f''(c) = \frac{f'(d) - f'(0)}{d - 0} = \frac{1}{d} \ .$$

As $d \in (0, 1)$, $1/d > 1$.

5) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Prove that f is decreasing on $[a, b]$ (i.e. $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$) if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.

Solution:

First suppose $f'(x) \leq 0$ for all $x \in (a, b)$. For $x_1 < x_2$, the Mean Value Theorem (applied to the interval $[x_1, x_2]$) implies there exists $c \in (x_1, x_2)$ such that $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \leq 0$. Therefore $f(x_2) - f(x_1) \leq 0$, i.e. $f(x_2) \leq f(x_1$, i.e. f is decreasing, as required.

Now assume that f is decreasing, and suppose (in order to obtain a contradiction) that $f'(d) > 0$ for some $d \in (a, b)$. Taking $\varepsilon = f'(d)/2 > 0$ in the definition of the derivative (as a limit), we see there exists $\delta > 0$ such that if $x > d$ with $x - d < \delta$ then

$$\left| \frac{f(x) - f(d)}{x - d} - f'(d) \right| < \frac{f'(d)}{2} \ .$$

Therefore $\frac{f(x) - f(d)}{x - d} > \frac{f'(d)}{2} > 0$, and hence $f(x) > f(d)$. Since $x > d$, this contradicts the fact that f is decreasing, as required.

(Note that this question resembles Theorem 2.4 (b), which states that if $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on $[a, b]$, i.e. $x_1 < x_2$ implies $f(x_1) > f(x_2)$.)

6) Suppose that f is continuous on $[0,1]$, differentiable on $(0,1)$, and $f(0)=0$. Prove that if f' is decreasing on $(0,1)$, then the function $g:(0,1)\to\mathbb{R}$ given by $g(x)=f(x)/x$ is decreasing on $(0,1)$.

Solution:

Since g is differentiable on $(0,1)$ it suffices, by question 5, to show that $g'(x)\leq 0$ for all $x\in(0,1)$. As

$$g'(x)=\frac{f'(x)x-f(x)}{x^2},$$

we only need to show that $f'(x)x-f(x)\leq 0$.

Applying the MVT to f on $[0,x]$, there exists $c\in(0,x)$ such that $f(x)-f(0)=f'(c)(x-0)$.

As f' is decreasing and $c<x$, then $f'(x)\leq f'(c)$. Therefore

$$f(x)=f'(c)x\geq f'(x)x$$

and hence $f'(x)x-f(x)\leq 0$ for all $x\in(0,1)$.