MTH5105 Differential and Integral Analysis 2012-2013

Solutions 4

1 Exercises for Feedback

1) Let the function $f\colon (0,\pi) \to \mathbb{R}$ be given by $x \mapsto \cos(x)$. What is $f((0,\pi))$?

   Show that f is invertible and that the inverse $g = f^{-1}$ is differentiable. Find a formula for the derivative g'.

   Compute the Taylor polynomial $T_{1,0}$ for g (recall that $T_{1,0}$ denotes the degree-one Taylor polynomial at the point 0). What is the remainder term in the Lagrange form?

   Hence show that for $x \in [0, 1/2]$,

$$|g(x) - \pi/2 + x| \leq \sqrt{3}/18 \approx 0.096 \ .$$

   Solution:

   The image $f((0,\pi))$ is the interval $(-1,1)$ (so $g = f^{-1}$ will be defined on this set).

   As $f'(x) = -\sin(x) < 0$ for all $x \in (0,\pi)$, f is strictly decreasing and therefore invertible, with differentiable inverse $g : (-1,1) \to \mathbb{R}$. (Of course we recognize from Calculus that $g = \arccos$.)

   To compute the derivative, we have $g'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$, and note that $f'(x) = -\sin(x) = -\sqrt{1-\cos^2(x)}$, therefore $g'(x) = -1/\sqrt{1-x^2}$.

   We have $g(0) = \pi/2$ and $g'(0) = -1$, so that $T_{1,0}(x) = \pi/2 - x$.

   From $g''(x) = -x(1-x^2)^{-3/2}$, the remainder term in the Lagrange form is given by

$$R = \frac{1}{2} g''(c) x^2 = -\frac{cx^2}{2(1-c^2)^{3/2}} \ .$$

   By Taylor's Theorem, there exists $c \in [0,x]$ such that $g(x) - T_{1,0}(x) = R$. For $x \leq 1/2$ we get the explicit estimate

$$|g(x) - \pi/2 + x| = |g(x) - T_{1,0}(x)| = |R| \leq \frac{|x|^3}{2(1-x^2)^{3/2}} \leq \frac{(1/2)^3}{2(1-1/4)^{3/2}} = \frac{1}{6\sqrt{3}} = \frac{\sqrt{3}}{18} \ .$$

2 Extra Exercises

2) Let $f(x) = \exp(1 - \exp(x)) = e^{1-e^x}$.

   (a) Determine the Taylor polynomials $T_{1,0}$, $T_{2,0}$, and $T_{3,0}$, of degrees 1, 2, and 3, respectively, for f at the point $a = 0$.

   (b) Prove that $f(x) > 1 - x$ for all $x > 0$.

Solution: (a) Using the chain and product rules, we compute

$$f'(x) = (-e^x)e^{1-e^x} = -e^{1+x-e^x} ,$$
$$f''(x) = -e^{1+x-e^x}(1-e^x) ,$$
$$f'''(x) = -e^{1+x-e^x}(-e^x) - e^{1+x-e^x}(1-e^x)^2 .$$

Therefore

$$f(0) = e^{1-e^0} = e^{1-1} = e^0 = 1 ,$$
$$f'(0) = -e^{1+0-1} = -1 ,$$
$$f''(0) = -e^{1+0-1}(1-1) = 0 ,$$
$$f'''(0) = -e^{1+0-1}(-1) - e^{1+0-1}(1-1)^2 = 1 .$$

Hence

$$T_{1,0}(x) = 1 - x ,$$
$$T_{2,0}(x) = 1 - x + 0x^2 = 1 - x ,$$
$$T_{3,0}(x) = 1 - x + \frac{x^3}{6} .$$

(b) Let $x > 0$. Taylor's Theorem tells us that $f(x) = T_{1,0}(x) + R_1$, where $R_1 = \frac{f''(c)}{2}x^2$ for some $c \in (0, x)$. In particular $c > 0$, so $e^c > 1$, therefore $f''(c) = -e^{1+c-e^c}(1-e^c) > 0$, hence $R_1 > 0$. Therefore $f(x) > T_{1,0}(x) = 1 - x$, as required.

3) In lectures we have shown that the number e can be expressed as

$$e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!} .$$

Show that the remainder term r_n in

$$n!e = n! \sum_{k=0}^{n} \frac{1}{k!} + r_n .$$

cannot be an integer. Hence deduce that e is irrational.

Solution:

We find

$$r_n = \sum_{k=n+1}^{\infty} \frac{n!}{k!} .$$

We estimate each term as

$$0 < \frac{n!}{k!} = \frac{1}{(n+1)(n+2)\ldots k} < \frac{1}{(n+1)^{k-n}}$$

so that

$$0 < r_n < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} = \sum_{l=1}^{\infty} \frac{1}{(n+1)^l} = \frac{1}{n} .$$

Hence

$$0 < r_n < \frac{1}{n} < 1 ,$$

and r_n therefore cannot be an integer.

Since $\sum_{k=0}^{n} \frac{n!}{k!}$ is an integer, we have shown that for all $n \in \mathbb{N}$, $n!e$ cannot be an integer. It follows that e cannot be a rational number, since if $e = p/q$ were rational then $n!q/qp$ would have to be an integer for n sufficiently large.