Coursework 1

<u>Ex 1</u> : $z_1 = 1+3i \qquad z_2 = 2-2i$

a) $$z_1 \cdot z_2 = (1+3i)(2-2i) = 2+6i-2i+6$$
$$= 8+4i$$

b) $$\frac{1}{z_1} = \frac{1}{1+3i} = \frac{1-3i}{1+9} = \frac{1}{10}(1-3i)$$

c) $$\frac{z_2}{z_1} = \frac{1}{10}(1-3i)(2-2i) = \frac{1}{5}(1-3i-i-3)$$
$$= -\frac{2}{5}(1+2i)$$

d) $$\frac{1}{z_1} + \frac{1}{z_2} = \frac{z_1+z_2}{z_1 z_2} = \frac{3+i}{4(2+i)} = \frac{1}{4}\frac{(3+i)(2-i)}{5}$$
$$= \frac{1}{20}(7-i)$$

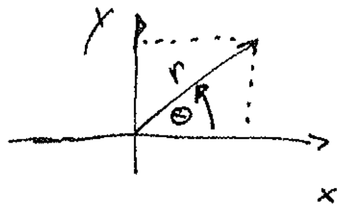
e) $$|z_1| = \sqrt{(1+3i)(1-3i)} = \sqrt{10}$$

f) $$|z_1 z_2| = \sqrt{64+16} = 4\sqrt{5}$$

g) $$\left|\frac{z_1 z_2}{z_1+z_2}\right| = \left|\frac{20}{7-i}\right| = 2\sqrt{2}$$

Ex 2 :

$$x + iy = r e^{i\theta}$$



a) $-2 = 2 e^{i\pi}$

b) $+i = e^{i\frac{1}{2}\pi} = e^{+i\frac{\pi}{2}}$

c) $z = +1 - i \Rightarrow r = \sqrt{1+1}$

$\theta = -\frac{1}{4}\pi \qquad \left(\cos\theta = \frac{1}{\sqrt{2}} \right)$

$\Rightarrow 1 - i = \sqrt{2}\, e^{-\frac{i\pi}{4}}$

d) $z = 1 + \sqrt{3}\, i$, $r = 2$, $\cos\theta = \frac{1}{2}$

$\Rightarrow z = 2 e^{i\frac{\pi}{3}}$

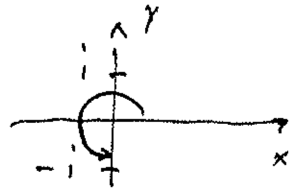
e) $z = \frac{1}{(1+i)^2}$

$1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$(1+i)^{-2} = \frac{1}{2} e^{-i\frac{\pi}{2}}$

*) 3.) $\quad (z - \frac{i}{2})^3 + i = 0$

$$\Rightarrow \quad (z - \tfrac{i}{2})^3 = -i = e^{\frac{3}{2}\pi i + 2\pi k i}, \quad k \in \mathbb{Z}$$



$-i = e^{-i\frac{\pi}{2}} = e^{+i\frac{3}{2}}$

$$\Rightarrow \quad z - \tfrac{i}{2} = e^{\frac{\pi}{2}i + \frac{2}{3}\pi k i}$$

2

get: $\quad z = \frac{i}{2} + e^{i\pi(\frac{1}{2} + \frac{2}{3}k)}, \quad k \in \mathbb{Z}$

Hence the three roots of the equation are:

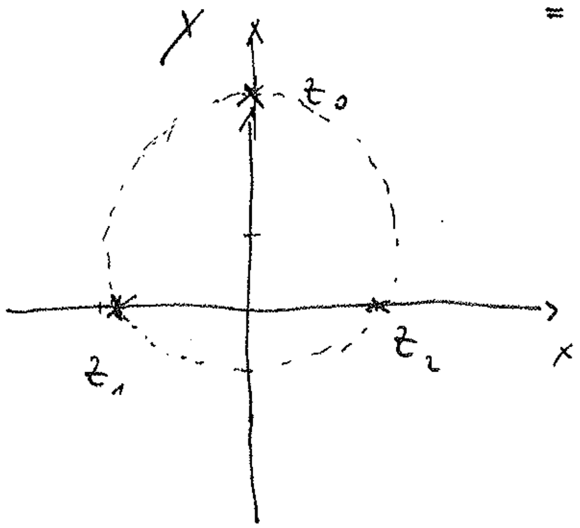
i) $k=0$: $\quad z_0 = \frac{i}{2} + e^{i\frac{\pi}{2}} = \frac{3}{2}i$ // 2

ii) $k=1$: $\quad z_1 = \frac{i}{2} + e^{i(\frac{\pi}{2} + \frac{2}{3}\pi)} = \frac{i}{2} + \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$ 2

$= \frac{i}{2} - \frac{\sqrt{3}}{2} - \frac{i}{2} = -\frac{\sqrt{3}}{2}$ //

iii) $k=2$: $\quad z_2 = \frac{i}{2} + e^{i\frac{11\pi}{6}} = \frac{i}{2} + \frac{\sqrt{3}}{2} - \frac{i}{2}$ 2

$= \frac{\sqrt{3}}{2}$ //



2

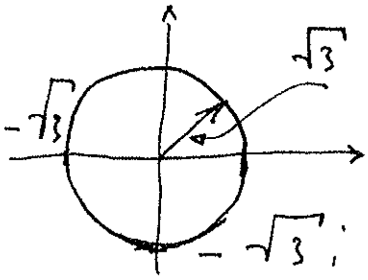
10

Ex 4: $z = x + iy$

a) $\frac{3}{\bar z} = z \implies \bar z z = 3 \implies |z|^2 = 3$

get $x^2 + y^2 = 3$ //

i.e. circle, radius $\sqrt{3}$



b) $\operatorname{Im}(z^3) > 0$

$z = r \cdot e^{i\theta} = r(\cos\theta + i\sin\theta)$

$\implies \operatorname{Im}(z^3) = \operatorname{Im}(r^3 e^{i3\theta})$

$= \operatorname{Im}(r^3(\cos 3\theta + i\sin 3\theta))$

$= r^3 \sin 3\theta$

$\implies \operatorname{Im}(z^3) > 0 \iff r^3 \sin 3\theta > 0$

$\iff r^3 > 0$ and $3\theta \in (0 + 2\pi k, \pi + 2\pi k$

$k \in \mathbb{Z}$

$\implies$ the set consists of all points $z = re^{i\theta}$

where $r > 0$,

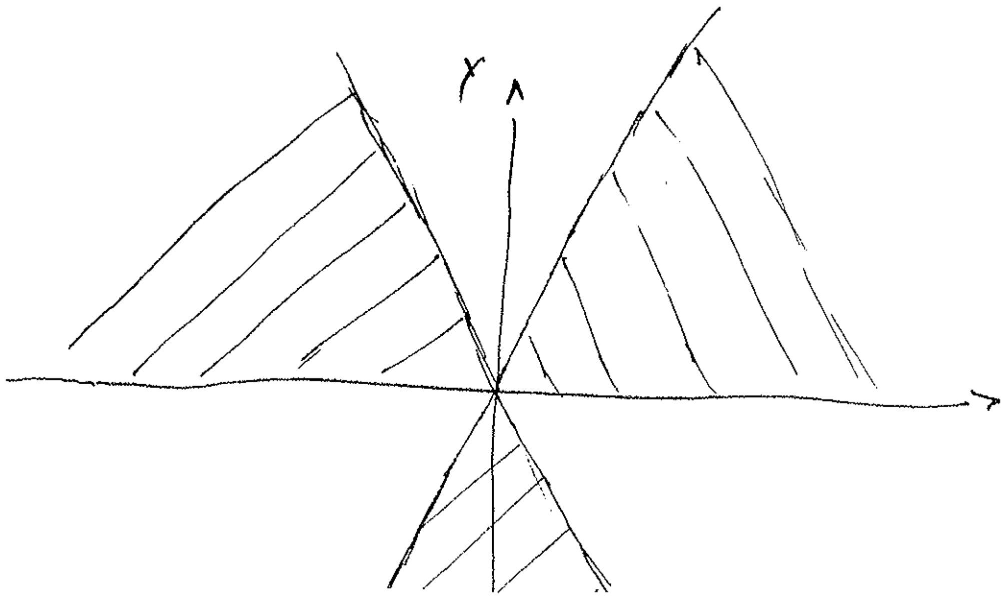
$\theta \in \left(0 + \frac{2}{3}\pi k, \frac{\pi}{3} + \frac{2}{3}\pi k\right)$

$k = 0, 1, 2$

$$\theta_0 \in \left(0, \frac{\pi}{3}\right)$$

$$\theta_1 \in \left(\frac{2}{3}\pi, \pi\right)$$

$$\theta_2 \in \left(\frac{4}{3}\pi, \frac{5}{3}\pi\right)$$



c) $|z| = \mathrm{Im}(z) + \frac{1}{2}$

$\iff (x^2+y^2)^{\frac{1}{2}} = y + \frac{1}{2}$

$\iff x^2 + y^2 = y^2 + y + \frac{1}{4}$

$\iff y = x^2 - \frac{1}{4}$ parabola

vertex at $\left(0, -\frac{1}{4}\right)$

or $z = -\frac{i}{4}$