1.) Euler's formula : $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ (*)

a) Let $z = e^{i\alpha}$ (i.e. choose $r=1$)

$\Rightarrow z^2 = (e^{i\alpha})^2 = e^{2i\alpha}$

<u>Using (*)</u> : $z^2 = (\cos\alpha + i\sin\alpha)^2$

$= \cos^2\alpha - \sin^2\alpha + 2i\sin\alpha\cos\alpha$

and $e^{2i\alpha} = \cos 2\alpha + i\sin 2\alpha$

Compare real and imaginary parts :

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

//

b) Similarly :

$z^3 = e^{3i\alpha} = (\cos\alpha + i\sin\alpha)^3$

$= \cos^3\alpha - 3\cos\alpha\sin^2\alpha + i(3\cos^2\alpha\sin\alpha - \sin^3\alpha)$

$\Rightarrow \cos 3\alpha = \cos^3\alpha - 3\cos\alpha\sin^2\alpha$

and $\sin 3\alpha = 3\cos^2\alpha\sin\alpha - \sin^3\alpha$ //

Further simplification possible using $\cos^2\alpha + \sin^2\alpha = 1$,

e.g. $\cos 3\alpha = \cos\alpha(4\cos^2\alpha - 3)$, $\sin 3\alpha = \sin\alpha(4\cos^2\alpha - 1)$, though not required

$$-1 = e^{\pi i + 2k\pi i}$$

2) a) $e^{2z} = -1$ $\qquad z = x + iy$

$\Leftrightarrow e^{2x} e^{2iy} = e^{i(\pi + 2k\pi)}$ $\qquad k \in \mathbb{Z}$

$\Leftrightarrow 2x = 0$ and $2y = (2k+1)\pi$

$\Leftrightarrow x = 0$ and $y = (k + \frac{1}{2})\pi$

solutions are: $z = i(k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$.

b) $\cosh z = 0$ $\qquad \cosh z = \frac{1}{2}(e^{z} + e^{-z})$

$\Leftrightarrow e^{z} + e^{-z} = 0$

$\Leftrightarrow e^{2z} + 1 = 0 \Leftrightarrow e^{2z} = -1$

using a): solutions are $z = i(k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$

c) $\cos z = 0$ $\qquad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\Leftrightarrow e^{iz} + e^{-iz} = 0 \Leftrightarrow e^{+2iz} = -1$

using a) $iz = i(k + \frac{1}{2}\pi)$

get: $z = (k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$

3.) Find $z \to w = \dfrac{az+b}{cz+d}$

such that $z \to w$:

$$0 \to i$$
$$-i \to 1$$
$$-1 \to 0$$

Get three equations:

$$0 = \frac{-a+b}{-c+d} \Rightarrow a = b$$

$$1 = \frac{-ia+b}{-ic+d} \Rightarrow d - ic = a(1-i)$$

$$i = \frac{b}{d} \Rightarrow b = di$$

$\Rightarrow$ express everything in terms of a, say:

$$d = -ia, \quad c = ia, \quad b = a$$

<u>Get</u>: $$w = -i \, \frac{z+1}{z-1}$$

Check: $$-i \, \frac{0+1}{0-1} = i \quad \checkmark$$

$$-i \, \frac{-i+1}{-i-1} = 1 \quad \checkmark$$

$$-i \, \frac{-1+1}{-1-1} = 0 \quad \checkmark$$

$z = x+iy$ and $w = u+iv$

4.) Consider $z \mapsto w = z^2 + z - 3$

a) What points are mapped to $v = c$?

$w = (x+iy)^2 + x + iy - 3 = x^2 - y^2 + 2ixy + x + iy + 3$

$\Rightarrow u = x^2 - y^2 + x - 3$ (*)

$v = 2xy + y$

Now: $v = c \Rightarrow c = y(2x+1)$

$\Leftrightarrow y = \dfrac{c}{2x+1}$ (hyperbola)

b) Find the image $\operatorname{Re}(z) = k$.

$\Rightarrow x = k$ substitute into (*), get

$u = k^2 - y^2 + k - 3$, $v = 2ky + y \Rightarrow y = \dfrac{v}{2k+1}$

get: $u = -\left(\dfrac{v}{2k+1}\right)^2 + k^2 + k - 3$ (parabola)

c) Sketch your results (from a) and b)) for $c = 1$ and $k = 0$