1.) a) $\lim\limits_{z \to -i} \dfrac{z^4}{z^3+i} = \dfrac{(-i)^4}{(-i)^3+i} = \dfrac{1}{2i} = -\dfrac{i}{2}$

b) $\lim\limits_{z \to \infty} \dfrac{(z-2i)^2(7z-3)}{(1-iz)^2(1+5z)}$

$= \lim\limits_{w \to 0} \dfrac{\left(\frac{1}{w}-2i\right)^2\left(\frac{7}{w}-3\right)}{\left(1-\frac{i}{w}\right)^2\left(1+\frac{5}{w}\right)}$

$= \lim\limits_{w \to 0} \dfrac{(1-2iw)^2(7-3w)}{(w-i)^2(w+5)}$

$= \dfrac{7}{(-i)^2 \cdot 5} = -\dfrac{7}{5}$

c) $\lim\limits_{z \to i+1} \left(\dfrac{z^2-2i}{z^5}\right)^{-1} = \left(\dfrac{(i+1)^2-2i}{(i+1)^5}\right)^{-1}$

$= \left(\dfrac{0}{(i+1)^5}\right)^{-1}$

$\Rightarrow \lim\limits_{z \to i+1} \dfrac{z^5}{z^2-2i} = \infty$

2.) a) $f(z) = 5z^4 + 7z^3 + 3\bar{z}$

Since $g(z) = z^n$, for $n \in \mathbb{Z} \setminus \{-1\}$,
and $h(z) = \bar{z}$ are continuous $\forall z \in \mathbb{C}$,
$f(z)$ is continuous by Prop. 1.10.

b) $f(z) = \dfrac{z - \bar{z}}{iz}$

i) $\underline{z \neq 0}$: as above, $z \mapsto z$ and
$z \mapsto \bar{z}$ are continuous for $\forall z \in \mathbb{C}$,
$f(z)$ is continuous $\forall z \in \mathbb{C} \setminus \{0\}$
by Prop. 1.10.

ii) <u>at the origin</u>: calculate limits
along e.g. real and imaginary axes

$$f(x) = \frac{x - x}{ix} = 0$$

$$f(y) = \frac{i\cancel{y} + i\cancel{y}}{i^2 \cancel{y}} = -2i \neq 0$$

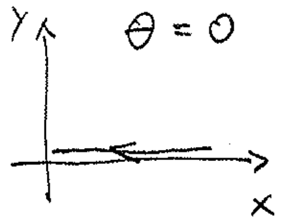
$\Rightarrow$ hence $\lim\limits_{z \to 0} f(z)$ does not exist.

3.) $\quad f(z) = \left(\dfrac{\bar{z}}{z} \right)^n \quad , \quad n \in \mathbb{N}$

Let $\quad z = r e^{i\theta}$

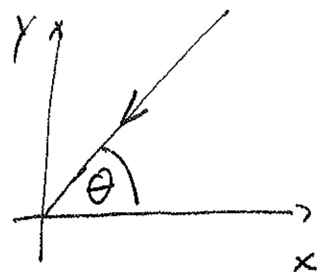
$$\Rightarrow \quad f(z) = \left(\frac{r e^{-i\theta}}{r e^{i\theta}} \right)^n = e^{-2i\theta n}$$

Now check different half-lines :

i) 

half-line with $\theta = 0$

$\Rightarrow \quad f(z) = 1$

ii) 

half-line with

$\theta = \dfrac{\pi}{2n}$

$\Rightarrow \quad f(z) = -1$

Hence, $\quad \lim\limits_{z \to 0} f(z)$ does not exist.

4.) a) Def. of derivative of $f(z)$ at z:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

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b) $f(z) = 4z^2 + \frac{i}{z}$

$$f'(-i) = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[4(-i+\Delta z)^2 - 4(-i)^2 + \frac{i}{-i+\Delta z} - \frac{i}{-i}\right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[-8i\Delta z + 4\Delta z^2 + i\left(\frac{1 - i(-i+\Delta z)}{-i+\Delta z}\right)\right]$$

$$= \lim_{\Delta z \to 0} \left[-8i + 4\Delta z + i\frac{-i}{-i+\Delta z}\right]$$

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$$= -8i + i = -7i$$

Or evaluate at z and substitute $z = -i$.

c) Binomial theorem: *

$$(a+b)^c = a^c + c\, b\, a^{c-1} + \dots + b^c$$

Here: $(z+\Delta z)^\alpha = z^\alpha + \alpha \Delta z\, z^{\alpha-1} + \dots + \Delta z^\alpha$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^\alpha - z_0^\alpha}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0^\alpha + \alpha \cdot \Delta z \cdot z_0^{\alpha-1} + \dots - z_0^\alpha}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\alpha \cdot z_0^{\alpha-1} + \Delta z \cdot \frac{\alpha(\alpha-1)}{2} z_0^{\alpha-2} + \dots\right)$$

$$= \alpha \cdot z_0^{\alpha-1}$$

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* This proof assumes α is an integer, although the result is true for any real α. In general the binomial series is infinite but proof still works.