1.) $f(z) = u(x,y) + i\,v(x,y)$

(1)

a) Cauchy-Riemann equations: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

b) i) $f = \cancel{k}z(xz+1) + ixy(1-2x) = xz^2 + z + ixy - 2ix^2y$

$= x(x^2-y^2+2ixy) + x + iy + ixy - 2ix^2y$

$\Leftrightarrow u = x^3 - xy^2 + x \,,\quad v = y + xy$

$\dfrac{\partial u}{\partial x} = 3x^2 - y^2 + 1 \,,\quad \dfrac{\partial u}{\partial y} = -2xy \,,\quad \dfrac{\partial v}{\partial x} = y \,,\quad \dfrac{\partial v}{\partial y} = 1 + x$

(4)

<u>CR</u>: $3x^2 - y^2 + \cancel{1} = \cancel{1} + x$ and $-2xy = -y$

case 1: $y = 0$, $3x^2 - x = (3x-1)x = 0$

case 2: $y \neq 0$, $x = \frac{1}{2} \Rightarrow \frac{3}{4} - \frac{1}{2} = y^2$, or $y^2 = \frac{1}{4}$

Hence set of 4 points: $z = 0$, $z = \frac{1}{3}$, $z = \frac{1}{2}(1 \pm i)$ //

ii) $f = x^2\left(\frac{1}{2} + i\right) + 2x + i(z - 2xy) = \frac{x^2}{2} + ix^2 + 2x + ix - y - 2ixy$

$\Leftrightarrow u = \frac{1}{2}x^2 + 2x - y$ and $v = x^2 + x - 2xy$

$\dfrac{\partial u}{\partial x} = x + 2 \,,\quad \dfrac{\partial u}{\partial y} = -1 \,,\quad \dfrac{\partial v}{\partial x} = 2x + 1 - 2y \,,\quad \dfrac{\partial v}{\partial y} = -2x$

<u>CR</u>: $x + 2 = -2x$ and $-1 = -2x - 1 + 2y$

get $x = -\frac{2}{3}$ and $y = x$ $\Rightarrow$ hence set of points:

$z = -\frac{2}{3}(1+i)$ //

(3)

c) Let $D = \{z : |z - z_0| < r\}$, where $r \in \mathbb{R}$.

Then the extra conditions on u and v are:

i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are defined everywhere in D

ii) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous at z_0

(2)

2) $f(z) = i e^{-iz}$, $z = x + iy$

a) $f(z) = i\left(e^{-ix} e^{y}\right) = i\left(\cos(-x) + i\sin(-x)\right)e^{y}$

$\Rightarrow u = e^{y}\sin x$, $v = \cos x \, e^{y}$

b) $\frac{\partial u}{\partial x} = e^{y}\cos x$, $\frac{\partial u}{\partial y} = e^{y}\sin x$

$\frac{\partial v}{\partial x} = -\sin x \, e^{y}$, $\frac{\partial v}{\partial y} = \cos x \, e^{y}$

<u>CR-equations</u>: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓ $(\forall (x,y))$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓ $(\forall (x,y))$

All derivatives exist, satisfy CR-equations, are continuous. $\Rightarrow f'(z)$ exists $\forall z \in \mathbb{C}$ (using Prop. 2.4).

c) $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^{y}\cos x + i(-\sin x \, e^{y})$

$= e^{y}(\cos x - i\sin x) = e^{y}e^{-ix} = e^{-iz}$

$\Rightarrow f'(z) = e^{-iz}$

<u>check</u>: $\frac{df}{dz} = \frac{d}{dz}\left(i e^{-iz}\right)$

$= e^{-iz}$ ✓

3) An entire function f is defined to be differentiable $\forall z \in \mathbb{C}$

$\Leftrightarrow$ the Cauchy-Riemann equations must hold everywhere in $\mathbb{C}$.

Now, $f(z) \in \mathbb{R} \quad \forall z \in \mathbb{C}$

that is $f(x+iy) = \big(u(x,y) + iv(x,y)\big) \in \mathbb{R}$

$\forall (x,y) \in \mathbb{R}^2$. Hence $v(x,y) = 0 \quad \forall (x,y) \in \mathbb{R}^2$

C-R :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = 0$$

$\forall (x,y) \in \mathbb{R}^2$

Thus $u = \text{const}$ for all $(x,y) \in \mathbb{R}^2$

$\Rightarrow f = u$ must be constant for all $z \in \mathbb{C}$.

4)

a) $f(z) = \sin|z|$

$\Rightarrow f(z) \in \mathbb{R} \ \forall z \in \mathbb{C}$, hence

f is not entire since it is not constant.

b) $f(z) = z^5 + iz^7 + z^{12}$

polynomial, differentiable $\forall z \in \mathbb{C}$

$\Rightarrow$ entire

c) $f(z) = x$

$f(z) \in \mathbb{R}$, not entire since not constant.

d) $f(z) = 2$

entire (proof above).