Queen Mary
University of London

MTH4102 Differential Equations
End of term test

Problem 1

$$y' = \frac{y^2 + 4x^2}{xy} = \frac{y}{x} + \frac{4x}{y}$$

Let $y(x) = xz(x)$, i.e.,

$$\begin{aligned}
& y' = z + xz' = z + \frac{4}{z} \\
\Rightarrow \quad & xz' = \frac{4}{z} \\
\Rightarrow \quad & \int z dz = 4 \int \frac{dx}{x} \\
\Rightarrow \quad & \frac{z^2}{2} = 4 \ln|x| + C \\
\Rightarrow \quad & z = \pm \left(8 \ln|x| + 2C\right)^{1/2} \\
\Rightarrow \quad & y = \pm x \left(8 \ln|x| + 2C\right)^{1/2}
\end{aligned}$$

Problem 2

$$0 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 \quad \Rightarrow \quad \lambda = 3$$

$$\begin{aligned}
y &= C_1 e^{3x} + C_2 x e^{3x} \\
y' &= 3C_1 e^{3x} + C_2 e^{3x} + 3C_2 x e^{3x}
\end{aligned}$$

$$\begin{aligned}
y(0) &= 1 = C_1 \\
y'(0) &= 1 = 3 + C_2 \quad \Rightarrow \quad C_2 = -2
\end{aligned}$$

$$y = e^{3x} - 2xe^{3x}$$

Problem 3

$$0 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) \quad \Rightarrow \quad \lambda_1 = 2 \neq 1, \quad \lambda_2 = 3 \neq 1$$

$$\begin{aligned}
y_p &= A \sin(x) + B \cos(x) \\
y_p' &= A \cos(x) - B \sin(x) \\
y_p'' &= -A \sin(x) - B \cos(x)
\end{aligned}$$

$$-A \sin(x) - B \cos(x) - 5A \cos(x) + 5B \sin(x) + 6A \sin(x) + 6B \cos(x) = \sin(x)$$

$$-A + 5B + 6A = 1 \quad \Rightarrow \quad 5A + 5B = 1$$
$$-B - 5A + 6B = 0 \quad \Rightarrow \quad A = B = \frac{1}{10}$$

$$y_p = \frac{1}{10}\sin(x) + \frac{1}{10}\cos(x)$$

Problem 4

$$0 = 2\lambda^2 + 3\lambda + 1 = (2\lambda + 1)(\lambda + 1) \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = -\frac{1}{2}$$

Since $f(x) = e^{-x/2}/2$, we must use variation of parameters.

$$y'' + \frac{3}{2}y' + \frac{1}{2}y = \frac{1}{2}e^{-x/2}$$

$$\begin{aligned} A(x) &= \frac{1}{2}\int dx \frac{e^{-x/2}e^{x/2}}{-1/2 + 1} = x \\ B(x) &= \frac{1}{2}\int dx \frac{e^{-x/2}e^{x}}{-1 + 1/2} = -\int e^{x/2}dx = -2e^{x/2} \end{aligned}$$

$$y_p = xe^{-x/2} - 2e^{-x/2}$$

Problem 5

$$\underline{\underline{A}} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda) = 0, \quad \lambda^{(A)} = 2, \quad \lambda^{(B)} = 1$$

$\lambda^{(A)} = 2$:

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} &2u_1 + u_2 = 2u_1 \\ &u_2 = 2u_2 \\ &\Rightarrow u_2 = 0 \end{aligned}$$

Choose $u_1 = 1$

$$\underline{u}^{(A)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda^{(B)} = 2$:

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} &2u_1 + u_2 = u_1 \\ &\Rightarrow -u_1 = u_2 \end{aligned}$$

Choose $u_1 = 1$

$$\underline{u}^{(B)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{y} = C^{(A)}e^{2x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C^{(B)}e^{x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$