1 Complex numbers and functions

Complex numbers

A complex number is an ordered pair (x, y) of real numbers. We denote the set of all complex numbers by $\mathbb{C}$ and we define "+" and "×" by:

$$(x,y) + (u,v) = (x+u, y+v)$$
$$(x,y) \times (u,v) = (xu - yv, xv + yu)$$

Note that with this definition of a complex number we avoid the mystery of "imaginary" numbers, and we could develop the whole of complex function theory using the notation of ordered pairs. However it is usual to write "$x+iy$" for the complex number (x,y). The rules for "+" and "×" now become:

$$(x+iy) + (u+iv) = (x+u) + i(y+v)$$
$$(x+iy) \times (u+iv) = (xu - yv) + i(xv + yu)$$

(which is just what we would expect from the usual rules of arithmetic, using $i \times i = -1$)

We shall use the notation "$x+iy$" from now on. Note that the real numbers $\mathbb{R}$ sit inside $\mathbb{C}$ as the numbers of the form $x + i0$, and also note that:

$$(0+i1) \times (0+i1) = (0-1) + i(0+0) = -1$$

In other words $i \times i = -1$ (something would have been seriously wrong with our theory if we got a different answer for $i \times i$!).

We write z for the complex number $x+iy$.

It is easy to check the usual rules for arithmetic (i.e. the axioms for a field):

Proposition 1.1 If $z_1, z_2, z_3 \in \mathbb{C}$ then

(i) $z_1 + z_2 \in \mathbb{C}$ & $z_1 z_2 \in \mathbb{C}$ (closure)

(ii) $z_1 + z_2 = z_2 + z_1$ & $z_1 z_2 = z_2 z_1$ (commutativity)

(iii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ $\quad z_1(z_2 z_3) = (z_1 z_2) z_3$ (associativity)

(iv) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (distributivity)

(v) $z_1 + 0 = z_1$ & $z_1 1 = z_1$ (zero & identity)

Write $z_1 = x_1 + iy_1$

(vi) $z_1 + (-z_1) = 0$ (where $-z_1 = -x_1 - iy_1$) and if $z_1 \neq 0$ then $z_1 z_1^{-1} = 1$, where $z_1^{-1} = \dfrac{x_1}{x_1^2 + y_1^2} - \dfrac{iy_1}{x_1^2 + y_1^2}$ (negative and inverse)

(The proofs all follow from the analogous properties of real numbers.) □

<u>Example</u> (of applying arithmetic operations)

$z_1 = 1+i$, $z_2 = 2-i \Rightarrow \begin{cases} z_1+z_2 = 3 & z_1 z_2 = 3+i \\ z_2^{-1} = \frac{1}{2-i} = \frac{2+i}{5} & \text{etc.} \end{cases}$

29.9.06

<u>Definitions</u> • If $z = x+iy$, x is called the <u>real part</u> $Re(z)$
& y " " " <u>imaginary part</u> $Im(z)$
(<u>Note</u> Some books call iy the imaginary part.)

• The <u>complex conjugate</u> of $z = x+iy$ is $\bar{z} = x-iy$

So: $\overline{(\bar{z})} = z$, $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$Re(z) = \frac{1}{2}(z+\bar{z})$ $\quad Im(z) = \frac{1}{2i}(z-\bar{z})$

• The <u>modulus</u> of $z = x+iy$ is $|z| = \sqrt{x^2+y^2}$

So: $|z| \geq 0$ $\quad (\& = 0 \iff z = 0)$

$|z| \geq Re(z)$ $\quad \& \ |z| \geq Im(z)$

Observe that $|z|^2 = z\bar{z}$, and so $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ (if $z \neq 0$)

<u>Proposition 1.2</u> If z, w are complex numbers, then

(i) $|zw| = |z||w|$

(ii) $|z+w| \leq |z|+|w|$

(iii) $|z-w| \geq \big||z|-|w|\big|$

<u>Proof</u> (i) $|zw|^2 = (zw)(\overline{zw}) = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2$

(ii) $|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w}) = z\bar{z}+z\bar{w}+\bar{z}w+w\bar{w}$

$= |z|^2 + 2Re(z\bar{w}) + |w|^2$

$\leq |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2$

$= (|z|+|w|)^2$

(iii) $|(z-w)+w| \leq |z-w|+|w|$ by (ii)

so $|z|-|w| \leq |z-w|$

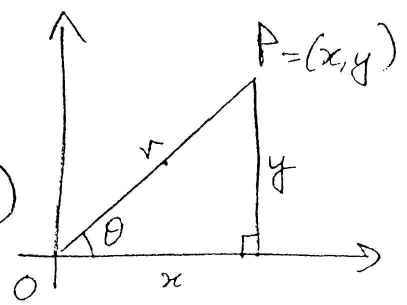
But similarly

$|w|-|z| \leq |w-z| = |z-w|$

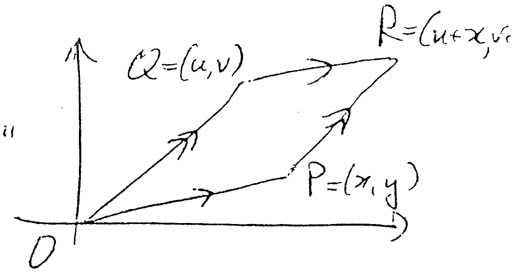
$\Big\}$ so $\big||z|-|w|\big| \leq |z-w|$

$\square$

Geometrical representation of complex numbers



We think of $x+iy$ as the point $P=(x,y)$ of the plane $\mathbb{R}^2$ (Gauss, Wessel, Argand). (Indeed this was our definition of a complex number.) Then $|z| = $ length $OP = r$, and addition of complex numbers is addition of vectors (the parallelogram rule).



Now 1.2(ii) is just the usual "triangle inequality" of Euclidean geometry.

Definition

~~The~~ An argument of $z(\neq 0)$ extended to arg(z), is ~~the~~ an angle θ such that $\cos\theta = \frac{x}{|z|}$ and $\sin\theta = \frac{y}{|z|}$. It is only defined up to the addition of multiples of 2π. We shall say that θ is the principal value of the argument if $0 \leq \theta < 2\pi$. (Note Some people say $-\pi < \theta \leq \pi$.)

We write $\mathrm{Arg}(z)$ for the principal value of an argument of z.

We can specify a complex number either by giving its real and imaginary parts, $z = x+iy$, or by giving its modulus and argument $z = r(\cos\theta + i\sin\theta)$. If we are adding complex numbers it is easier to use the $x+iy$ form, but for multiplying it is easier to use the modulus and argument form.

Proposition 1.3 Let $z, w \in \mathbb{C}$ (both non-zero). Then

$$|zw| = |z||w| \quad \text{and} \quad \arg(zw) = \arg z + \arg w \quad (\text{modulo } 2\pi)$$

Proof (We already have a proof that $|zw| = |z||w|$ but we shall have another proof here.) Let $\theta = \arg z$ and $\phi = \arg w$

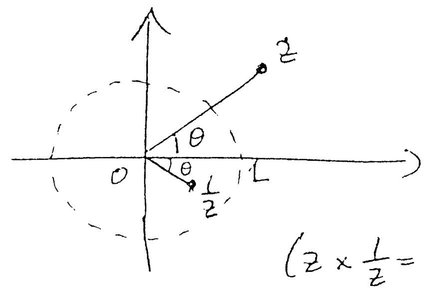
$$\begin{aligned} zw &= |z|(\cos\theta + i\sin\theta)\,|w|(\cos\phi + i\sin\phi) \\ &= |z||w|\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\big) \\ &= |z||w|\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \end{aligned}$$

But this is in "modulus & argument" form, so $|zw| = |z||w|$ and $\theta+\phi$ is an argument of zw. $\square$

So the geometric rule for multiplying complex numbers is "multiply moduli and add arguments".

<u>Example 1</u> To define $\frac{1}{z}$ geometrically:
(let $w = z$, then use $|wz| = |w|\,|z|$, $\arg(wz) = \arg(w) + \arg(z)$)

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \quad \text{and} \quad \arg\left(\frac{1}{z}\right) = -\arg z.$$



$\left(z \times \frac{1}{z} = 1\right)$

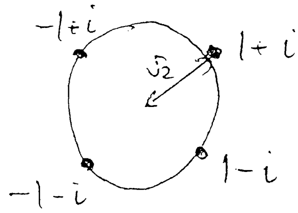
<u>Example 2</u> To find all the nth roots of a given $w \in \mathbb{C}$.

$$z^n = w \iff |z|^n = |w| \ \& \ n \arg z = \arg w \pmod{2\pi} \quad (\text{by } 1.3)$$

$\iff |z|$ is the real positive nth root of $|w|$ and $n \arg z = \arg w + 2k\pi$ (some $k \in \mathbb{Z}$)

$$\iff |z| = |w|^{1/n} \ \& \ \arg z = \frac{1}{n}\arg w + \frac{2k\pi}{n} \quad (k \in \mathbb{Z})$$

As we let k run from 0 to $n-1$ we get n distinct solutions for z, evenly spaced around a circle centre O, radius $|w|^{1/n}$. For example the 4th roots of -4 are:



$w = -4$ has $|w| = 4$, $\arg w = \pi$

So 4th roots z have $|z| = 4^{1/4} = \sqrt{2}$

and $\arg z = \frac{\pi}{4} + \frac{2k\pi}{4} = \frac{\pi}{4} + \frac{k\pi}{2} \quad (k = 0, 1, 2, 3)$

<u>Theorem 1.4</u> (Euler, 1760) $\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$

(where $e^{i\theta}$ denotes the sum of the power series $1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$)

This is called <u>Euler's formula</u>. It is one of the most amazing formulae in mathematics. For example if we set $\theta = \pi$ it says $e^{\pi i} = -1$, linking the most most important mathematical quantities $(-1, i, \pi, e)$ in one formula. We cannot <u>prove</u> this formula till later in the course when we have looked at properties of power series, but you should already be able to see that the real part of $e^{i\theta}$ is $1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots$, the Taylor series for $\cos\theta$, and the imaginary part of $e^{i\theta}$ is $\frac{\theta}{1!}-\frac{\theta^3}{3!}+\dots$, the Taylor series for $\sin\theta$: these two observations can be developed into a formal proof.

However we shall use (1.4) as a convenient notation, even before we have proved it. Thus if $z \neq 0$, with $|z|=r$ and $\arg z = \theta$ we shall often write $z = re^{i\theta}$. We call this the <u>polar form</u> of z.

This notation suggests that the following should be true:

$$e^{i\theta} = e^{i(\theta+2n\pi)} \quad (n\in\mathbb{Z})$$ — by Euler's formula

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

$$1/e^{i\theta} = e^{-i\theta}$$ — by 'familiar properties' of exp.

$$(e^{i\theta})^n = e^{in\theta} \quad (n\in\mathbb{Z})$$ — de Moivre's Thm

These are all true, and easily proved using (1.4) but they are not "obvious" ($e^{i\theta}$ <u>means</u> $1+\frac{i\theta}{1!}+\frac{(i\theta)^2}{2!}+\dots$; "raising the real number e to the imaginary power $i\theta$" is of course meaningless!)

2.10.06

Functions of a complex variable

Let $S \subset \mathbb{C}$. A *function* f on S is a rule which assigns to each $z \in S$ a complex number $f(z)$, called the value of f at z.

Examples

- *Polynomials*; e.g. $f(z) = z^2$ or $f(z) = 1 + iz - z^2 + (1+i)z^3$; these are defined $\forall z \in \mathbb{C}$
- *Rational functions*: one polynomial divided by another, e.g. $f(z) = \frac{z}{z^2+2}$; these are defined everywhere except at roots z of the the denominator.
- *The exponential function*: $f(z) = e^z$, which is defined to be the sum $1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$ (or equally well, by (1.4) as $e^x(\cos y + i \sin y)$). We shall see later that the series $1 + \frac{z}{1!} + \dots$ converges (abs.) $\forall z \in \mathbb{C}$, so e^z is defined $\forall z \in \mathbb{C}$.
- *Trigonometric and hyperbolic functions*: For complex z we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

(this agrees with the definition for z *real*).

Similarly, by analogy with the real case, we define

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

Note that $\cos z + i \sin z = e^{iz} = \cosh iz + \sinh iz$

Given any complex function f we can write $z = x + iy$ and also write $f(z)$ as $\underbrace{u(x,y)}_{\text{real part of } f(z)} + i\underbrace{v(x,y)}_{\text{imaginary part of } f(z)}$. where $u, v \in \mathbb{R}^2$, $u \to v$

Thus we think of f as a map $\mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (u(x,y), v(x,y))$

not done

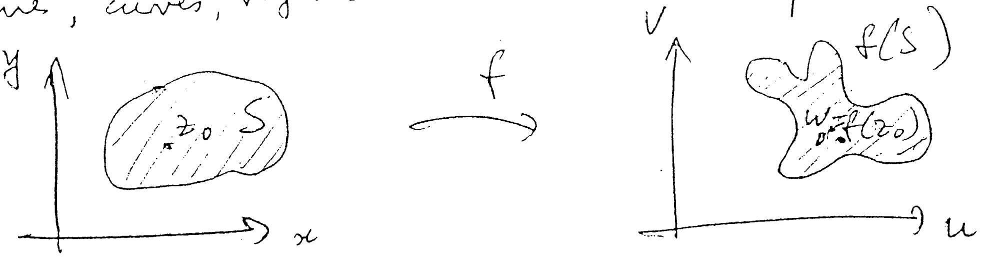
Examples

1) $f(z) = z^2 = (x^2 - y^2) + i(2xy)$
2) $f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$
3) $f(z) = e^z = e^x \cos y + i e^x \sin y$

in each case the first summand is $u(x,y)$ and the second is $iv(x,y)$

Transformations of the complex plane

A complex function is a map (transformation) $\mathbb{C} \to \mathbb{C}$. But to draw a graph of such a map we would need $2+2=4$ real dimensions. So instead we shall examine what happens to various lines, curves, regions under such a map.



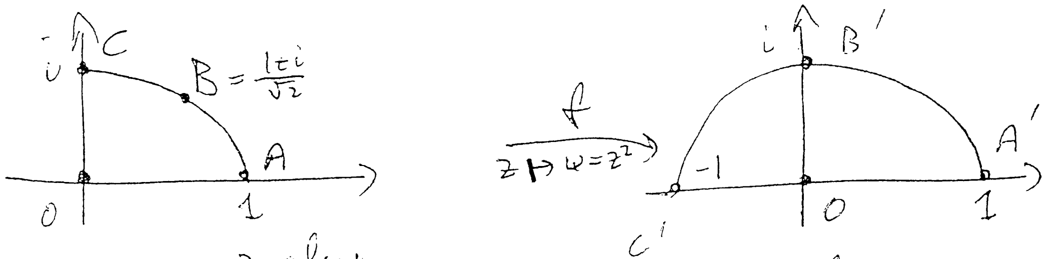
z-plane w-plane

Example $w = z^2$

In (x,y), (u,v) coordinates this is $u + iv = (x^2-y^2) + 2ixy$

i.e. $u = x^2 - y^2 \quad v = 2xy$

It is easier to understand this map if we write z and w in polar coordinates, since then it sends $z = re^{i\theta}$ to $w = r^2 e^{2i\theta}$

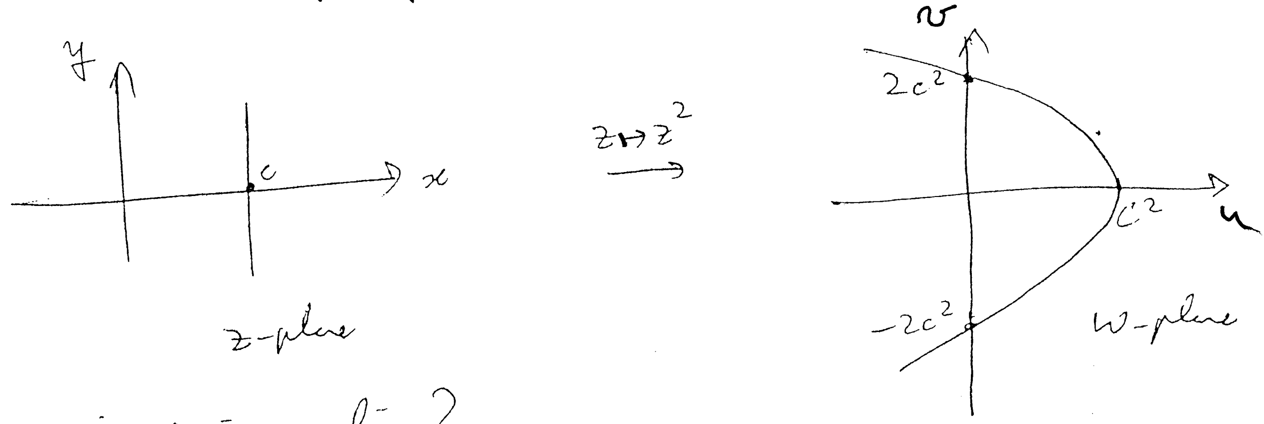


z-plane w-plane

real axis	$\longrightarrow$	positive real axis
imaginary axis	$\longrightarrow$	negative real axis
quarter circle radius r centre O	$\longrightarrow$	half circle radius r^2 centre O
straight line through origin at angle θ	$\longrightarrow$	straight half-line starting at origin at angle 2θ

Question 1

What is the image of a straight line $x = c$ under the transformation $z \mapsto z^2$?



Equation of image line?

$z \mapsto z^2$ is the map $(x,y) \mapsto (u,v)$ where $u = x^2 - y^2$, $v = 2xy$

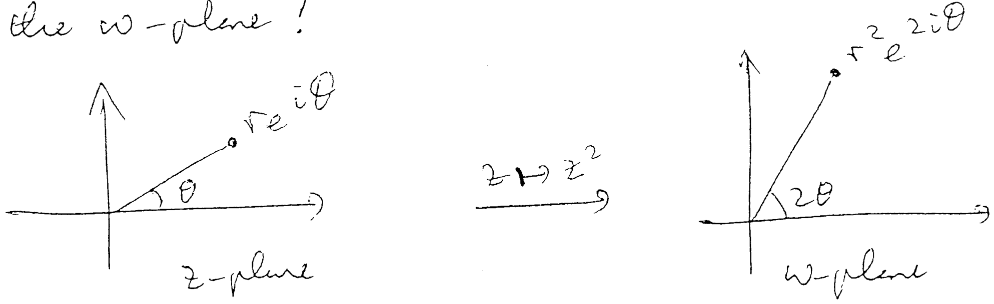
Setting $x = c$ we get $u = c^2 - y^2$, $v = 2cy$

Eliminating y, the equation of the image line is

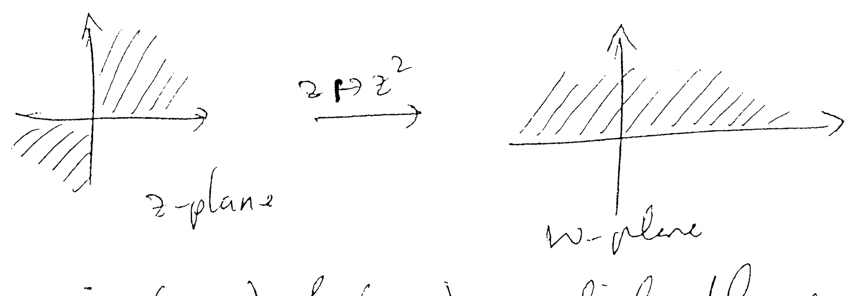
$$u = c^2 - \frac{v^2}{4c^2} \qquad \text{(a parabola)}$$

Question 2

Under $z \mapsto z^2$ what region(s) of the z-plane map to the upper half of the w-plane?



$$
\begin{aligned}
w = r^2 e^{2i\theta} \in \text{upper } \tfrac{1}{2}\text{-plane} &\iff 0 + 2k\pi \le 2\theta \le \pi + 2k\pi \quad (k \in \mathbb{Z}) \\
&\iff k\pi \le \theta \le k\pi + \tfrac{\pi}{2} \quad (\text{" "}) \\
&\iff z \in \text{1st or 3rd quadrant}
\end{aligned}
$$



Alternatively, using (x,y) & (u,v) coordinates, the answer to question 2 is that

$$
\begin{aligned}
w \in \text{upper } \tfrac{1}{2}\text{-plane} &\iff v \ge 0 \\
&\iff 2xy \ge 0 \\
&\iff x \ge 0 \ \& \ y \ge 0 \ \text{or} \ x \le 0 \ \& \ y \le 0 \\
&\iff z \in \text{1st or 3rd quadrant.}
\end{aligned}
$$

Proposition 1.5 The following kinds of transformation all send
(i) straight lines to straight lines, and
(ii) circles to circles:

1) translations : $w = z + c$ (c a complex constant)
2) rotations : $w = e^{i\theta} z$ (θ a real constant)
3) dilations : $w = rz$ ($r \neq 0$ a real constant)
4) linear maps : $w = \lambda z + c$ ($\lambda (\neq 0)$ complex, c complex constant)
5) complex conjugation : $w = \bar{z}$

Proof 1), 2), 3) are all obvious geometrically, as is 5) since $z \to w = \bar{z}$ is just reflection in the x-axis $((x,y) \mapsto (x,-y))$.
The easiest way to prove 4) is to write $\lambda = re^{i\theta}$. Then $z \mapsto \lambda z + c$ is the composite:

$$z \xmapsto{\text{dil.}} rz \xmapsto{\text{rot.}} re^{i\theta} z \xmapsto{\text{tra}} re^{i\theta} z + c$$

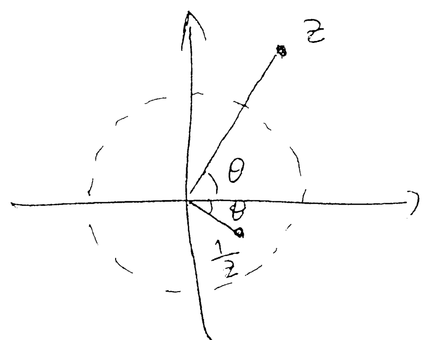
So the result follows as 1), 2) & 3) send straight lines to straight lines and circles to circles. □

Another function we often need to consider is $z \to w = \frac{1}{z}$. We can regard this as a composition:

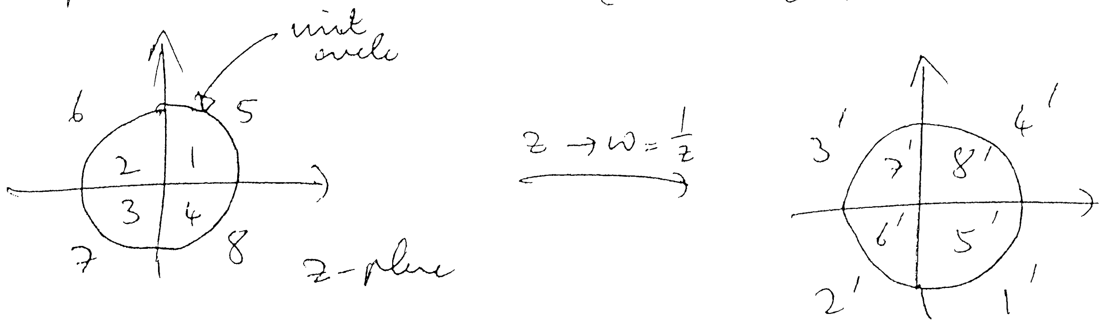
$$z \mapsto \frac{z}{|z|^2} \longmapsto \frac{\bar{z}}{|z|^2} = \frac{1}{z}$$

"inversion" in unit circle

reflection in real axis



$z \mapsto w = \frac{1}{z}$ maps $\mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ (or $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$) as follows:



<u>Proposition 1.6</u> $z \mapsto w = \frac{1}{z}$ maps

(i) circles not through 0 to circles not through 0
(ii) circles through 0 to straight lines not through 0
(iii) straight lines not through 0 to circles through 0
(iv) straight lines through 0 to straight lines through 0

[So if we regard a straight line as a "circle passing through ∞" this Proposition says $z \mapsto w = \frac{1}{z}$ sends "circles to circles".]

<u>Proof</u> Let our circle (or line) be

$$a(x^2+y^2) + bx + cy + d = 0 \qquad (a, b, c, d \in \mathbb{R})$$

If $a = 0$ this is a straight line.
If $a \neq 0$ it is a circle $\left(\text{centre } \left(-\frac{b}{2a}, -\frac{c}{2a}\right), \text{ radius } \sqrt{\frac{b^2+c^2-4ad}{2a}}\right)$.
In either case it passes through $0 \iff d = 0$.

Now let $w = \frac{1}{z}$, so $z = \frac{1}{w}$ i.e. $x + iy = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$.
Substituting for x and y our equation becomes

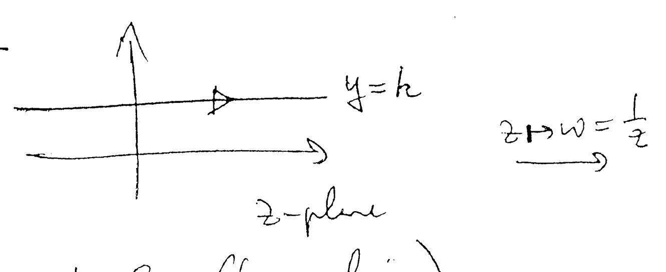
$$a\left(\frac{u^2+v^2}{(u^2+v^2)^2}\right) + \frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} + d = 0$$

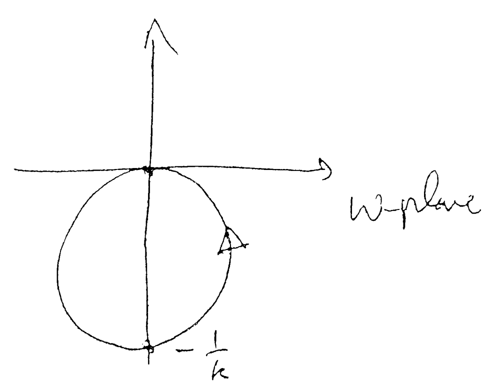
that is,

$$d(u^2+v^2) + bu - cv + a = 0$$

This is a circle $\iff d \neq 0$, a line $\iff d = 0$, and passes through the origin $\iff a = 0$. $\square$

<u>Example</u>





$y - k = 0$ (horiz. line)

Substitute $x = \frac{u}{u^2+v^2}$ $y = -\frac{v}{u^2+v^2}$

Get $\frac{v}{u^2+v^2} + k = 0$ i.e. $k(u^2+v^2) + v = 0$

circle centre $\left(0, -\frac{1}{2k}\right)$, radius $\frac{1}{2k}$

Definition A map of the form $z \mapsto w = \frac{az+b}{cz+d}$ $(a,b,c,d \in \mathbb{C},\ ad-bc \neq 0)$ is called a fractional linear or Möbius transformation.

Any Möbius transformation can be written as a composition:

$$z \mapsto Z \mapsto W \mapsto w$$

where $Z = cz+d$ (if $c \neq 0$) [if $c=0$ the Möb tra is linear: $z \to az+b$]

$W = \frac{1}{Z}$ and $w = \alpha W + \beta$

with $\alpha \cdot \frac{1}{cz+d} + \beta = \frac{az+b}{cz+d}$ i.e. $\beta c = a$ & $\beta d + \alpha = b$

i.e. $\beta = \frac{a}{c}$ & $\alpha = b - \frac{ad}{c}$

It follows from (1.5) & (1.6) that

Proposition 1.7 Möbius transformations send lines and circles to lines and circles.

Comments

1. $z \to w = \frac{az+b}{cz+d}$ can be extended to a map $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$ by sending $z = \infty$ to $w = \frac{a}{c}$ and sending $z = -\frac{d}{c}$ to $w = \infty$. It is a bijection $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ and has inverse $w \mapsto z = \frac{dw-b}{-cw+a}$.

2. By an easy calculation the composite of two Möbius transformations is given by multiplying the corresponding matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

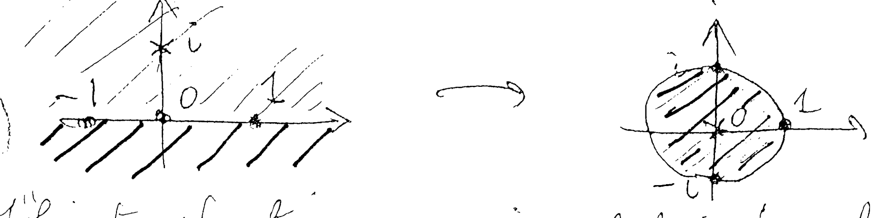
3. Given any 3 distinct points z_1, z_2, z_3 in the z-plane and any 3 distinct points w_1, w_2, w_3 in the w-plane, there exist a unique Möbius transformation sending $z_1 \to w_1$, $z_2 \to w_2$, $z_3 \to w_3$.

Example Find a Möbius tra. sending $-1 \to i$, $0 \to 1$ and $1 \to -i$.

We need $i = \frac{-a+b}{-c+d}$ $\quad 1 = \frac{b}{d}$ $\quad -i = \frac{a+b}{c+d}$

So $b = d$ & $\left.\begin{array}{l} a-b-ic+ib=0 \\ a+b+ic+ib=0 \end{array}\right\}$ $\therefore$ $2a+2ib=0$ $\therefore$ $a=-ib$ & $b=-ic$ so $c=ib$

Hence the transformation is $z \to \frac{-iz+1}{iz+1} = \frac{+z+i}{-z+i}$



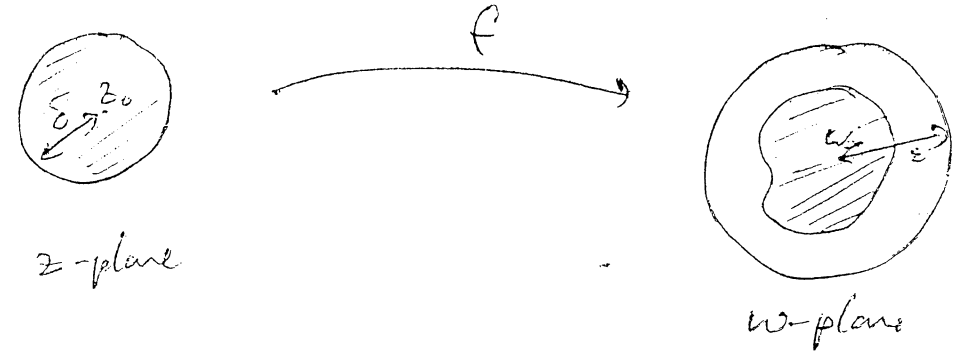
real axis $\to$ unit circle (unique circle through $-i, 1, i$)

[N.B Möbius transformations are very important in hyperbolic geometry]

Limits and Continuity

Let f be a complex function. We say that $\lim_{z \to z_0} f(z) = w_0$ if "when z is close to z_0, $f(z)$ is close to w_0". Formally

Definition $\lim_{z \to z_0} f(z) = w_0$ if given any real $\varepsilon > 0$ $\exists$ real $\delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ for all z with $0 < |z - z_0| < \delta$.



["However small an $\varepsilon > 0$ we are given, we can find some value of δ such that $f(z)$ is ε-close to w_0 whenever z is δ-close to z_0."]

Notes i) We do not require that $f(z_0)$ itself be defined

2) When a limit exists it is unique (we cannot have $f(z)$ arbitrarily close to two <u>different</u> points w_0 as $z \to z_0$).

Examples

1) $f(z) = \bar{z}/2$; $\lim_{z \to z_0} \bar{z}/2 = \bar{z}_0/2$ (Given any $\varepsilon > 0$, take $\delta = \varepsilon$; then $|z - z_0| < \delta \Rightarrow \left|\frac{\bar{z}}{2} - \frac{\bar{z}_0}{2}\right| < \varepsilon$; in fact $\delta = 2\varepsilon$ would do here)

2) $f(z) = \bar{z}/z$; $\lim_{z \to 0} \bar{z}/z$ does not exist. (For real $z \neq 0$, $\bar{z}/z = 1$ and for imaginary $z \neq 0$, $\bar{z}/z = -1$; so $\lim_{z \to 0} \bar{z}/z$ cannot exist)

N.B For $z_0 \neq 0$ $\lim_{z \to z_0} \bar{z}/z = \bar{z}_0/z_0$

The technique used in 2) is a common way of showing a limit does not exist: we show that we get two different limits as $z \to z_0$ along two different directions.

Example

3) Show that $\lim_{z \to 0} z^2/\bar{z}^2$ does not exist.

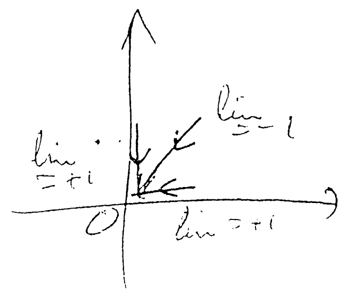
Here $z = x$ real gives $z^2/\bar{z}^2 = x^2/x^2 = 1$

and $z = iy$ gives $\dfrac{(iy)^2}{\overline{(iy)}^2} = 1$

but $z = (1+i)t$ (t real) gives

$$z^2/\bar{z}^2 = \frac{2it}{-2it} = -1$$

So $\lim_{z \to 0} z^2/\bar{z}^2$ does not exist.



__Definition__ f is __continuous__ at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$
(i.e. ~~if~~ limit exists & is equal to $f(z_0)$).

__Examples__ 1) $f(z) = az + b$ is continuous at all points $z_0 \in \mathbb{C}$

2) $f(z) = \begin{cases} \bar{z}/z & z \neq 0 \\ 1 & z = 0 \end{cases}$ is not continuous at $z_0 = 0$ (but is continuous elsewhere)

3) $f(z) = \begin{cases} z^2/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}$ is continuous at all $z_0 \in \mathbb{C}$ $\left(\lim_{z \to 0} \frac{z^2}{|z|} = 0 \right)$

If we write $f(z) = u(x,y) + iv(x,y)$ then continuity of f is expressible in terms of continuity of the functions u and v:
Consider $z = x+iy$ as the point $(x,y) \in \mathbb{R}^2$ and

__Proposition 1.8__ Let $f(z) = u(x,y) + iv(x,y)$. Then

$$\lim_{z \to z_0} f(z) = u_0 + iv_0 \iff \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \ \& \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$$

(complex defn. of limit)

(real defn. of limit)

Let: $z = x+iy$, $z_0 = x_0 + iy_0 \Rightarrow$ distance from (x,y) to (x_0,y_0) in $\mathbb{R}^2$ is $|z - z_0|$

__Proof__ $\Rightarrow$: Given any $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall z$ with $0 < |z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$

i.e. $|(u(x,y) + iv(x,y)) - (u_0 + iv_0)| < \varepsilon$

So $|u(x,y) - u_0| < \varepsilon$ & $|v(x,y) - v_0| < \varepsilon$

But $|z - z_0| = d((x,y),(x_0,y_0))$ the distance from (x,y) to (x_0,y_0) in $\mathbb{R}^2$ so we have proved that $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

$\Leftarrow$: Take $\varepsilon > 0$, then because u & v have (real) limit u_0 & v_0 as $(x,y) \to (x_0, y_0)$ we know that $\exists\, \delta_1, \delta_2 > 0$ s.t.

$$0 < d((x,y),(x_0,y_0)) < \delta_1 \Rightarrow |u(x,y) - u_0| < \varepsilon/2$$
& $$0 < d((x,y),(x_0,y_0)) < \delta_2 \Rightarrow |v(x,y) - v_0| < \varepsilon/2$$

Let δ be the smaller of δ_1, δ_2. Then

$$0 < d((x,y),(x_0,y_0)) < \delta \Rightarrow |u(x,y) - u_0 + i(v(x,y) - v_0)| < \varepsilon/2 + \varepsilon/2$$

(by 1.2)

i.e. $0 < |z - z_0| < \delta \Rightarrow |f(z) - (u_0 + iv_0)| < \varepsilon$, as required. $\square$

Using 1.8 and standard results of real limits we have:

<u>Prop. 1.9</u> If $f(z)$, $g(z)$ have limits w_0, w_1 as $z \to z_0$ then
(i) $f(z) + g(z)$ has limit $w_0 + w_1$ as $z \to z_0$
(ii) $f(z)g(z)$ " " $w_0 w_1$ " " "
(iii) $1/f(z)$ has limit $1/w_0$ as $z \to z_0$ (provided $w_0 \neq 0$) $\square$

and so

<u>Proposition 1.10</u> If $f(z)$, $g(z)$ are continuous at z_0 then so are
(i) $f(z) + g(z)$
(ii) $f(z)g(z)$
(iii) $1/f(z)$ (provided $f(z_0) \neq 0$) $\square$

It is obvious that a constant function $f(z) = c$ is continuous $\forall z \in \mathbb{C}$ and also that the identity function $f(z) = z$ is continuous for all $z \in \mathbb{C}$ (just take $\delta = \varepsilon$ in defn. of limit). So we deduce from 1.10 that

<u>Prop 1.11</u> (i) Every polynomial $f(z) = a_0 + \dots + a_n z^n$ with coeffs $a_0, \dots a_n$ in $\mathbb{C}$, is continuous $\forall z \in \mathbb{C}$
(ii) Every rational function $f(z) = \dfrac{p(z)}{q(z)}$ (p, q polynomials) is continuous at all $z \in \mathbb{C}$ except possibly at the roots of $q(z)$. $\square$

<u>Example</u> $$\lim_{z\to i} \frac{z^2+2i}{z^2+4i} = \lim_{z\to i}(z^2+2i) \times \lim_{z\to i}\frac{1}{z^2+4i} \qquad (1.9\,(ii))$$
$$= \left(\left(\lim_{z\to i} z\right)^2 + 2i\right) \Big/ \left(\lim_{z\to i} z\right)^2 + 4i \qquad (1.9\,(i), (ii)\ \&\ (iii))$$
$$= (i^2+2i)/(i^2+4i) \quad (\text{using continuity of } f(z) = z)$$

Hence $\frac{z^2+2i}{z^2+4i}$ is continuous at $z_0 = i$

It is convenient to extend the definitions of limit and continuity to include the point "infinity" :-

<u>Definition</u>

$\lim_{z\to\infty} f(z) = w_0 \iff$ given $\varepsilon>0$ $\exists R>0$ s.t. $|f(z)-w_0|<\varepsilon$ whenever $|z|>R$

$\lim_{z\to z_0} f(z) = \infty \iff$ given $R>0$ $\exists \delta>0$ s.t. $|f(z)|>R$ whenever $0<|z-z_0|<\delta$

$\lim_{z\to\infty} f(z) = \infty \iff$ given $R>0$ $\exists S>0$ s.t. $|f(z)|>R$ whenever $|z|>S$.

Now rational functions can be regarded as continuous <u>everywhere</u> on $\mathbb{C}\cup\{\infty\}$.

In practice one of the **best** ways of dealing with "∞" in limits is to use the substitution $z=\frac{1}{Z}$ and let $Z\to 0$:-

<u>Examples</u>

1) $\displaystyle \lim_{z\to\infty}\frac{5z+i}{z+i} = \lim_{Z\to 0}\frac{5/Z+i}{1/Z+i} = \lim_{Z\to 0}\frac{5+iZ}{1+iZ} = 5$

2) $\displaystyle \lim_{z\to -1}\frac{iz+3}{z+1} = \infty$ since $\displaystyle \lim_{z\to -1}\frac{z+1}{iz+3} = \frac{0}{3-i} = 0$