1 Complex numbers and functions

Complex numbers

A complex number is an ordered pair (x,y) of real numbers. We denote the set of all complex numbers by $\mathbb{C}$ and we define "+" and "×" by

$$(x,y) + (u,v) = (x+u,\ y+v)$$
$$(x,y) \times (u,v) = (xu - yv,\ xv + yu)$$

Note that with this definition of a complex number we avoid the mystery of "imaginary" numbers, and we could develop the whole of complex function theory using the notation of ordered pairs. However it is usual to write "$x+iy$" for the complex number (x,y). The rules for "+" and "×" now become:

$$(x+iy) + (u+iv) = (x+u) + i(y+v)$$
$$(x+iy) \times (u+iv) = (xu - yv) + i(xv+yu)$$

which is just what we would expect from the usual rules of arithmetic using $i^2 = -1$. (i: Euler, 1777)

$$(x+iy)(u+iv) = xu + ixv + iyu + i^2 yv$$

We shall use the notation "$x+iy$" from now on. Note that the real numbers $\mathbb{R}$ sit inside $\mathbb{C}$ as the numbers of the form $x = x+i0$ and also note that $i = 0+i1$ satisfies indeed

$$(0+i1)\times(0+i1) = (0-1) + i(0-0) = -1$$

We write z for the complex number $x+iy$.
It is easy to check the usual rules for arithmetic
(i.e. the axioms for a field):

<u>Proposition 1.1</u> If $z_1, z_2, z_3 \in \mathbb{C}$ then

(i) $z_1 + z_2 \in \mathbb{C}$ & $z_1 z_2 \in \mathbb{C}$ (closure)

(ii) $z_1 + z_2 = z_2 + z_1$ & $z_1 z_2 = z_2 z_1$ (commutativity)

(iii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ & $z_1(z_2 z_3) = (z_1 z_2) z_3$ (associativity)

(iv) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (distributivity)

(v) $z_1 + 0 = z_1$ & $z_1 1 = z_1$ (zero and identity)

(vi) $z_1 + (-z_1) = 0$ (where $-z_1 = -x_1 - iy_1$) (negative and inverse)

and if $z_1 \neq 0$ then $z_1 z_1^{-1} = 1$ where

$$z_1^{-1} = \frac{x_1}{x_1^2 + y_1^2} - \frac{i y_1}{x_1^2 + y_1^2}$$

(The proofs all follow from the analogous properties of real numbers) □

<u>Example</u> (of applying arithmetic operations)

$$z_1 = 1+i, \quad z_2 = 2-i \quad \rightsquigarrow \quad \begin{cases} z_1 + z_2 = 3 & z_1 z_2 = 3+i \\ z_2^{-1} = \dfrac{1}{2-i} = \dfrac{2+i}{5} & \text{etc} \end{cases}$$

Definitions • If $z = x+iy$, x is called the real part $Re(z)$
& y is called the imaginary part $Im(z)$
(Note: some books call iy the imaginary part)

• The complex conjugate of $z = x+iy$ is $\bar{z} = x-iy$

So: $\overline{(\bar{z})} = z$, $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

$Re(z) = \frac{1}{2}(z+\bar{z})$, $Im(z) = \frac{1}{2i}(z-\bar{z})$

• The modulus (absolute value) of $z = x+iy$ is $|z| = \sqrt{x^2+y^2}$

So: $|z| \geq 0$ $(|z| = 0 \Leftrightarrow z = 0)$

$|z| \geq |Re(z)|$, $|z| \geq |Im(z)|$

Observe that $|z|^2 = z\bar{z}$ and therefore $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ (if $z \neq 0$)

Proposition 1.2 If z, w are complex numbers, then

(i) $|zw| = |z||w|$

(ii) $|z+w| \leq |z| + |w|$

(iii) $|z-w| \geq \big||z| - |w|\big|$

Proof (i) $|zw|^2 = (zw)\overline{(zw)} = zw\bar{z}\bar{w} = z\bar{z}\,w\bar{w} = |z|^2|w|^2$

(ii) $|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$

$= |z|^2 + 2Re(z\bar{w}) + |w|^2$

$\leq |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2$

(iii) $|(z-w)+w| \leq |z-w| + |w|$ by (ii)

$$\left.\begin{aligned} &\text{so } |z| - |w| \leq |z-w| \\ &|(w-z)+z| \leq |w-z| + |z| \\ &\text{so } |w| - |z| \leq |w-z| \end{aligned}\right\} \rightsquigarrow \big||z|-|w|\big| \leq |z-w|$$

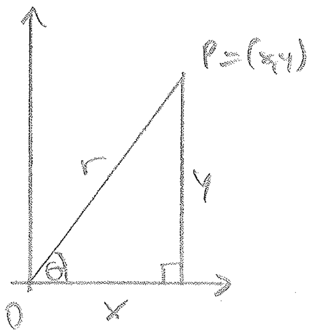
□

Geometrical representation of complex numbers

We think of $x+iy$ as the point $P=(x,y)$

of the plane $\mathbb{R}^2$ (Wessel 1797, Argand 1806, Gauß 1831)

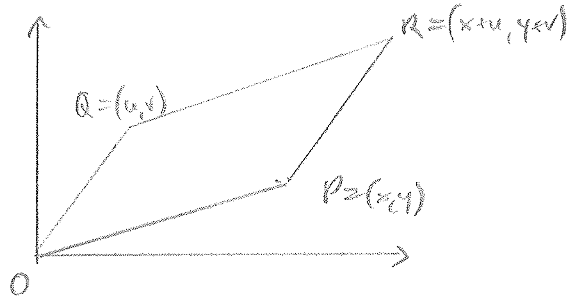
(Indeed, this was our definition of a complex number)



Then $|z| = $ length $OP = r$, and

addition of complex numbers is addition of vectors

(the parallelogram rule).



Now 1.2(ii) is just the usual "triangle inequality" of Euclidean geometry.

Definition The argument of $z\,(\neq 0)$ is the angle θ such that $\cos\theta = \frac{x}{|z|}$ and $\sin\theta = \frac{y}{|z|}$. (It is only defined up to multiples of 2π.) We shall say that θ is the principal value of the argument if $0 \leq \theta < 2\pi$.

(Note: some people choose $-\pi < \theta \leq \pi$.)

We can specify a complex number either by giving its real and imaginary parts, $z = x+iy$, or by giving its modulus and argument, $z = r(\cos\theta + i\sin\theta)$. If we are adding complex numbers it is easier to use the $x+iy$ form, but for multiplying it is easier to use the modulus and argument form:

Proposition 1.3 Let $z, w \in \mathbb{C}$ (both non-zero) Then

$$|zw| = |z|\,|w| \quad \text{and} \quad \arg(zw) = \arg(z) + \arg(w) \quad (\text{modulo } 2\pi)$$

Proof (We already have a proof that $|zw| = |z||w|$, but we shall have another proof here) Let $\theta = \arg z$ and $\phi = \arg w$.

$$zw = |z|(\cos\theta + i\sin\theta)\,|w|(\cos\phi + i\sin\phi)$$

$$= |z||w|\Big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\Big)$$

$$= |z||w|\Big(\cos(\theta+\phi) + i\sin(\theta+\phi)\Big)$$

is a "modulus & argument" form, therefore $|zw| = |z||w|$

and $\arg(zw) = \theta + \phi$ $\square$

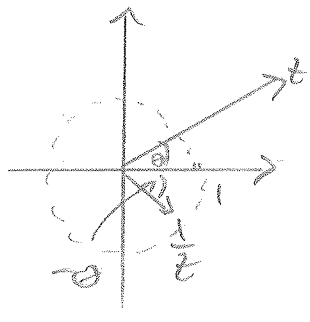
Notation: $\arg(z)$ argument of z, $\operatorname{Arg}(z)$ principal argument of z

So the general rule for multiplying complex numbers is "multiply moduli and add arguments."

Example 1 To define $\frac{1}{z}$ geometrically: since $z \cdot \frac{1}{z} = 1$

$$|z|\left|\frac{1}{z}\right| = \left|z\cdot\frac{1}{z}\right| = 1 \quad \text{and} \quad \arg(z) + \arg\left(\frac{1}{z}\right) = \arg\left(z\cdot\frac{1}{z}\right) = 0$$

$$\leadsto \left|\frac{1}{z}\right| = \frac{1}{|z|} \quad \text{and} \quad \arg\left(\frac{1}{z}\right) = -\arg z$$



Example 2 To find all the nth roots of a given $w \in \mathbb{C}$:

$z^n = w \iff |z|^n = |w|$ & $n \arg z = \arg w$ (by 1.3)

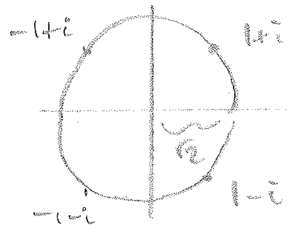
$\iff |z|$ is the real positive nth root of $|w|$ and $n\arg z = \arg w + 2k\pi$ $(k \in \mathbb{Z})$

$\iff |z| = |w|^{1/n}$ and $\arg z = \frac{1}{n}\arg w + \frac{k}{n}2\pi$ $(k \in \mathbb{Z})$

As we let k run from 0 to $n-1$ we get n distinct solutions for z, evenly spaced around a circle with centre at 0, radius $|w|^{1/n}$. For example, the 4th roots of -4:

$w=-4 \rightarrow |w|=4$, $\arg w = \pi$

$|z| = 4^{1/4} = \sqrt{2}$, $\arg z = \frac{\pi}{4} + \frac{k}{4}2\pi = \frac{\pi}{4} + \frac{k\pi}{2}$ $(k=0,1,2,3)$



e.g. $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$

1.10.(b)

<u>Theorem 1.4</u> (Euler formula 1748, proved by Cotes 1714)

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

(where $e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$ as power series).

This is called <u>Euler's formula</u>. It is one of the most amazing formulae in mathematics. For example, if we set $\theta=\pi$ it says $e^{i\pi}=-1$, linking the five most important mathematical quantities $(0, 1, i, \pi, e)$ in one formula: $\underline{e^{i\pi}+1=0}$

We cannot <u>prove</u> this formula yet (need properties of power series), but we can observe that

$\mathrm{Re}\left(e^{i\pi}\right) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots = \cos\theta$ and

$\mathrm{Im}\left(e^{i\pi}\right) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots = \sin\theta$

This can be used for a formal proof.

However, we shall use (1.4) as a convenient notation, even before we have proved it. Thus, if $z \neq 0$, with $|z| = r$ and $\arg z = \theta$ we shall often write $z = re^{i\theta}$. We call this the <u>polar form</u> of z

This notation suggests that the following should be true.

$$e^{i\theta} = e^{i(\theta + 2n\pi)} \qquad n \in \mathbb{Z} \qquad \text{(Euler formula)}$$

$$\left.\begin{aligned} & e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)} \\ & 1/e^{i\theta} = e^{-i\theta} \\ & (e^{i\theta})^n = e^{in\theta} \qquad n \in \mathbb{Z} \end{aligned}\right\} \text{ properties of exp}$$

These are all true, and easily proved using (1.4) but they are not "obvious" ($e^{i\theta}$ <u>means</u> $1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \dots$, $(\text{real})^{\text{imaginary}}$ is otherwise meaningless!)

back to the 4-th roots of -4:

$$z^4 = -4 \quad : \quad -4 = 4e^{i\pi} = 4e^{i(\pi + 2k\pi)}$$

$$z = \left(4e^{i(\pi + 2k\pi)}\right)^{1/4} = 2^{1/2} e^{i\frac{2k+1}{4}\pi}$$

$$z_0 = 2^{1/2} e^{i\pi/4} = 2^{1/2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) = 1+i$$

$$z_1 = 2^{1/2} e^{i3\pi/4} = 2^{1/2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right) = -1+i$$

$$z_2 = -1-i, \quad z_3 = 1-i, \quad z_4 = z_0$$

Functions of a complex variable

Let $S \subset \mathbb{C}$. A function f on S is a rule which assigns to each $z \in S$ a complex number $f(z)$, called the value of f at z.

Examples

- Polynomials: e.g. z^2 or $1 + iz - z^2 + (1+i)z^3$

  These are defined for all $z \in \mathbb{C}$

- Rational functions: one polynomial divided by another, e.g. $\dfrac{z}{z^2+2}$

  These are defined everywhere except at roots z of the denominator $(\pm\sqrt{2}\,i)$

- The exponential function: e^z, which is defined to be the sum

  $1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dots$ (or, equally well, by (1.4) as $e^x(\cos y + i \sin y)$)

  We shall see later that the series converges absolutely for all $z \in \mathbb{C}$, so e^z is defined for all $z \in \mathbb{C}$.

- Trigonometric and hyperbolic functions: For complex z we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

(this agrees with the definition for z real)

Similarly, by analogy with the real case, we define

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

Note that $\cos z + i \sin z = e^{iz} = \cosh iz + \sinh iz$

Given any complex function f, we can write $z = x+iy$

and also write $f(z) = \underbrace{u(x,y)}_{\text{real part of } f(z)} + i \underbrace{v(x,y)}_{\text{imaginary part of } f(z)}$

Thus, we think of f as a map $\mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (u,v)$

4.10

Examples: $\quad f(z) = u(x,y) + i\, v(x,y)$

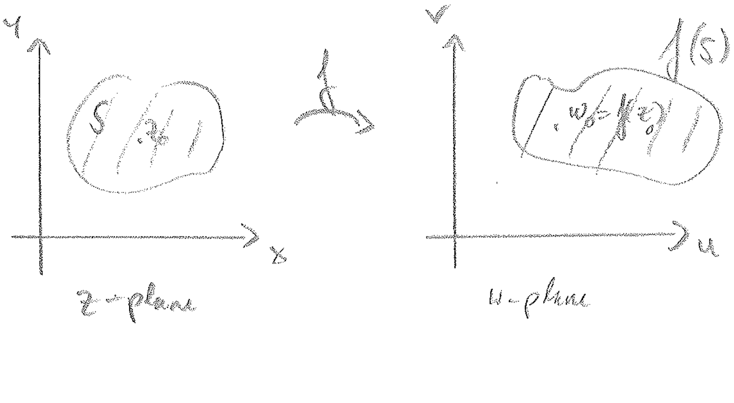
1) $f(z) = z^2 = (x^2 - y^2) + i\,(2xy)$

2) $f(z) = \dfrac{1}{z} = \dfrac{x}{x^2+y^2} + i\, \dfrac{-y}{x^2+y^2}$

3) $f(z) = e^z = e^x \cos y + i\, e^x \sin y$

Transformations of the complex plane

A complex function is a map (transformation) $\mathbb{C} \to \mathbb{C}$. But to draw a graph of such a map we would need $2+2=4$ real dimensions. So instead we shall examine what happens to various lines, curves, regions under such a map.

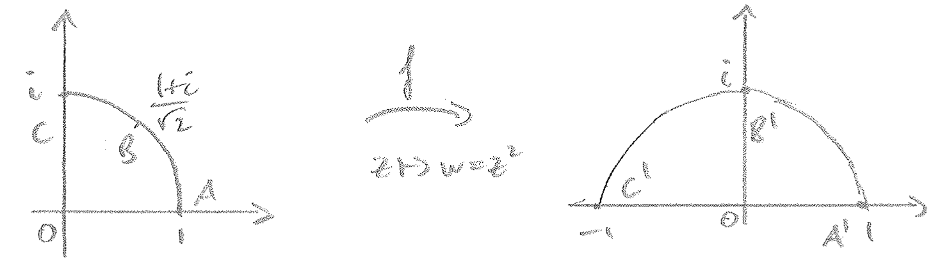


<u>Example</u> $w = z^2$

In (x,y), (u,v) coordinates this is $u + iv = (x^2 - y^2) + i(2xy)$

i.e. $u = x^2 - y^2$, $v = 2xy$. We'll return to this below.

More easily, in polar coordinates $z = re^{i\theta}$ gives $w = r^2 e^{2i\theta}$



z-Plane w-Plane

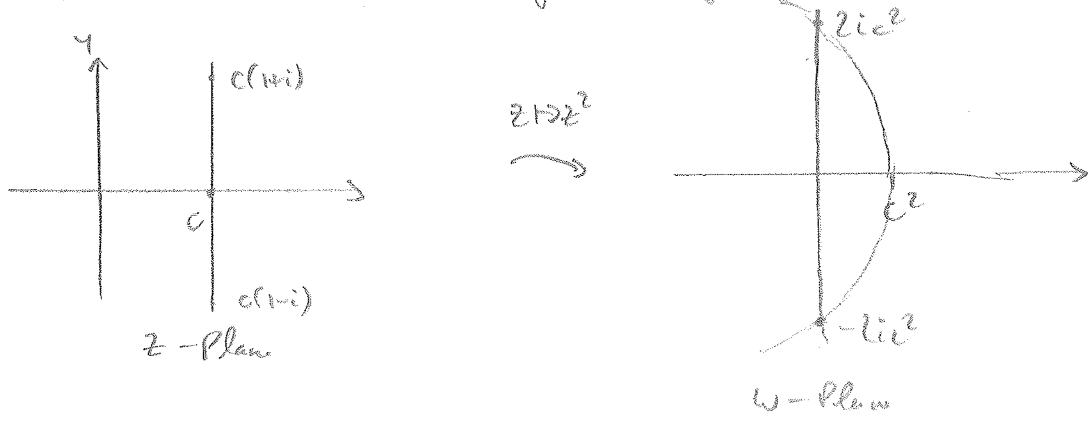
real axis $\mathbb{R}$ $\rightarrow$ non-negative real axis $\mathbb{R}_0^+$

imaginary axis $i\mathbb{R}$ $\rightarrow$ non-positive real axis $\mathbb{R}_0^-$

quarter circle radius r centre O $\rightarrow$ half circle radius r^2 centre O

straight line through origin at angle θ $\rightarrow$ straight half-line starting at origin at angle 2θ

<u>Question 1</u>: What is the image of a straight vertical line at $x = c$?



z-Plane w-Plane

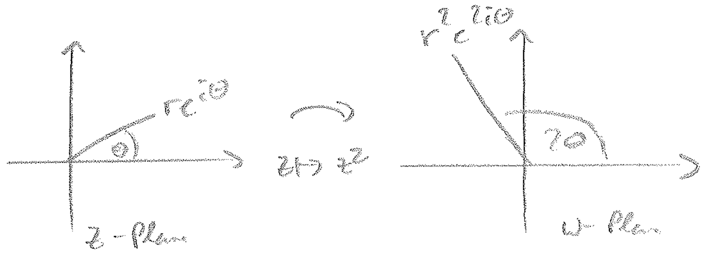
Equation of image line?

$z \mapsto z^2$ is the map $(x,y) \mapsto (u,v)$ with $u = x^2 - y^2$, $v = 2xy$

Setting $x = c$ fixed we get $u = c^2 - y^2$, $v = 2cy$

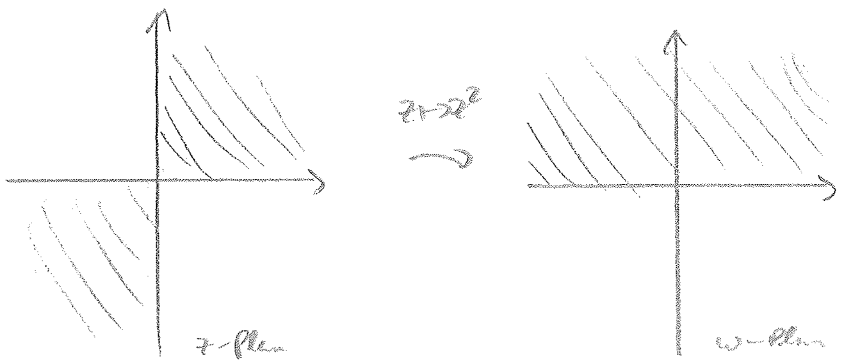
eliminating y, we get $u = c^2 - \left(\frac{v}{2c}\right)^2 = c^2 - \frac{1}{4c^2}v^2$ parabola

Question 2: Under $z \mapsto z^2$ what region(s) of the z-plane map to the upper half of the w-plane?



$w = r^2 e^{2i\theta} \in$ upper half plane $\Leftrightarrow 0 + 2k\pi \le 2\theta \le \pi + 2k\pi$ $(k \in \mathbb{Z}$

$\Leftrightarrow k\pi \le \theta \le k\pi + \frac{\pi}{2} \Leftrightarrow z \in$ first or third quadrant



Alternatively, using (x,y) & (u,v) coordinates, the answer is

w in upper half plane $\Leftrightarrow v \geq 0 \Leftrightarrow 2xy \geq 0$

$\Leftrightarrow x \geq 0$ & $y \geq 0$ or $x \leq 0$ & $y \leq 0$

$\Leftrightarrow z \in$ first or third quadrant

Proposition 1.5 The following kinds of transformations all send

(i) straight lines to straight lines, and

(ii) circles to circles:

1) translations $w = z + c$ (c a complex constant)

2) rotations $w = e^{i\theta} z$ (θ a real constant)

3) dilations $w = rz$ ($r \neq 0$ a real constant)

4) linear maps $w = \lambda z + c$ ($\lambda \neq 0$ complex, c complex constants)

5) complex conjugation $w = \bar{z}$

Proof 1), 2), 3) all obvious geometrically,

4) with $\lambda = re^{i\theta}$

$z \mapsto \lambda z + c$ is composed of

$$z \underset{3)}{\mapsto} rz \underset{2)}{\mapsto} re^{i\theta} z \underset{1)}{\to} re^{i\theta} z + c$$

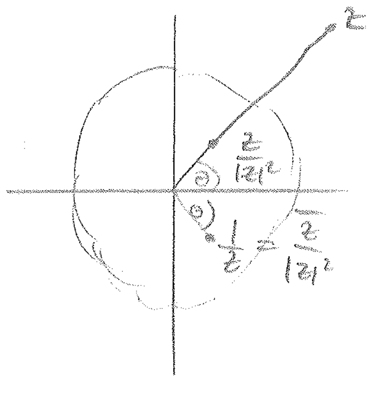
5) $z = x + iy \mapsto w = x - iy$ reflection at x-axis. $\square$

Another function we often need to consider is $z \mapsto w = \frac{1}{z}$. We can regard this as a composition:

$$z \mapsto \frac{z}{|z|^2} \mapsto \frac{\bar{z}}{|\bar{z}|^2} = \frac{1}{z}$$

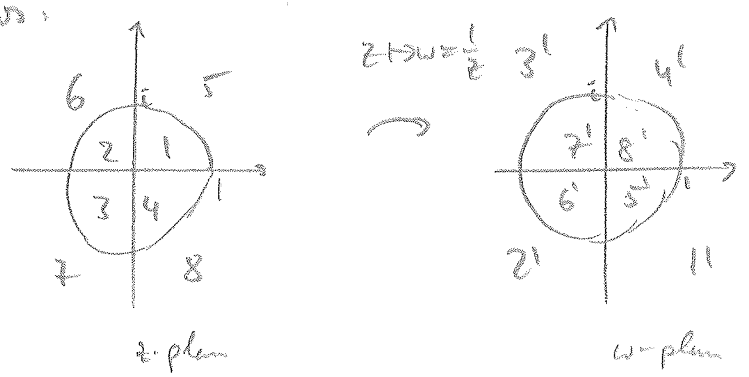
"inversion" in unit circle

reflection at real axis



$z \mapsto w = \frac{1}{z}$ maps $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$
(or $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$)

as follows:



Proposition 1.6 $\quad z \mapsto w = \frac{1}{z}$ maps

(i) circles not through 0 to circles not through 0

(ii) circles through 0 to straight lines not through 0

(iii) straight lines not through 0 to circles through 0

(iv) straight lines through 0 to straight lines through 0

[Therefore, if we regard a straight line as a "circle passing through ∞",

this proposition says $z \mapsto w = \frac{1}{z}$ map "circles to circles"]

Proof $\quad$ Let our circle (or line) be

$$a(x^2+y^2) + bx + cy + d = 0 \qquad (a, b, c, d \in \mathbb{R})$$

If $a = 0$ this is a straight line (b and c may not both be zero)

If $a \neq 0$ this is a circle (center $\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$, radius $\sqrt{\frac{b^2+c^2-4ad}{2a}}$)

In either case it passes through 0 $\iff d = 0$

Now let $w = \frac{1}{z}$, so $z = \frac{1}{w}$

i.e. $x+iy = \frac{u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2}$

Substituting for x and y the equation becomes

$$a\,\frac{u^2+v^2}{(u^2+v^2)^2} + b\,\frac{u}{u^2+v^2} - c\,\frac{v}{u^2+v^2} + d = 0$$

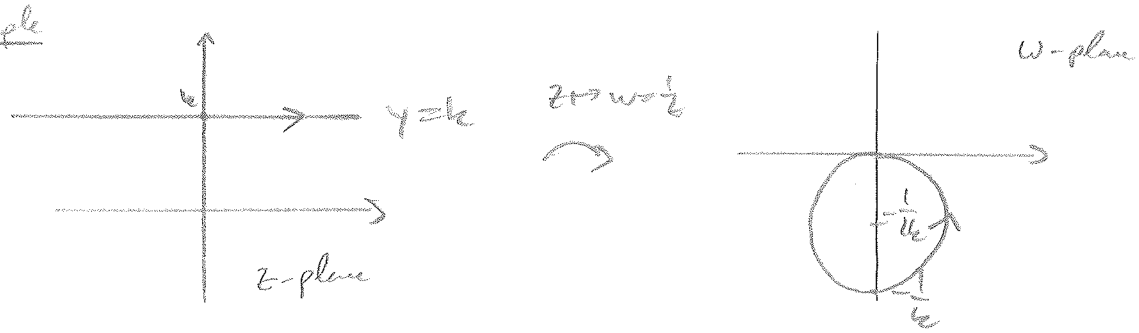
That is, $d(u^2+v^2) + bu - cv + a = 0$

$d \neq 0 \iff$ circle

$d = 0 \iff$ line (b and c not both zero)

$a = 0 \iff$ passes through O $\square$

<u>Example</u>



$y - k = 0$ horizontal line

Substitute $x = \frac{u}{u^2+v^2}$ $y = \frac{-v}{u^2+v^2}$

get $\frac{-v}{u^2+v^2} - k = 0$, i.e. $k(u^2+v^2) + v = 0$

circle centred at $(0, -\frac{1}{2k})$, radius $\frac{1}{2k}$

<u>Definition</u> A map of the form $z \mapsto w = \frac{az+b}{cz+d}$

($a, b, c, d \in \mathbb{C}$, $ad-bc \neq 0$) is called a <u>fractional linear</u> or

<u>Möbius transformation</u>.

Any Möbius transformation (with $c \neq 0$) can be written as a composition

$$z \to Z \to W \to w$$

where $Z = cz+d$, $W = \frac{1}{Z}$, $w = \alpha W + \beta$

with $\alpha \frac{1}{cz+d} + \beta = \frac{az+b}{cz+d}$, i.e. $\beta c = a$ & $\beta d + \alpha = b$

i.e. $\beta = \frac{a}{c}$ & $\alpha = b - \frac{ad}{c}$

($c = 0$: $w = \frac{az+b}{d}$ linear)

It follows from (1.5) & (1.6) that

<u>Proposition 1.7.</u> Möbius transformations send lines and circles to lines and circles.

<u>Comments</u> 1) $z \mapsto w = \frac{az+b}{cz+d}$ can be extended to a map

$\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by sending $z = \infty$ to $w = \frac{a}{c}$

and sending $z = -\frac{d}{c}$ to $w = \infty$. The extension is a

<u>bijection</u> with inverse $w \mapsto z = \frac{dw-b}{-cw+a}$

2) By an easy calculation the composite of two Möbius transformations

is given by multiplying the corresponding matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(Note: $ad - bc \neq 0 \Leftrightarrow \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$)

3) Given any 3 distinct points z_1, z_2, z_3 in the z-plane and any 3 distinct points w_1, w_2, w_3 in the w-plane, there exists a unique Möbius transformation sending $z_1 \to w_1$, $z_2 \to w_2$, $z_3 \to w_3$.

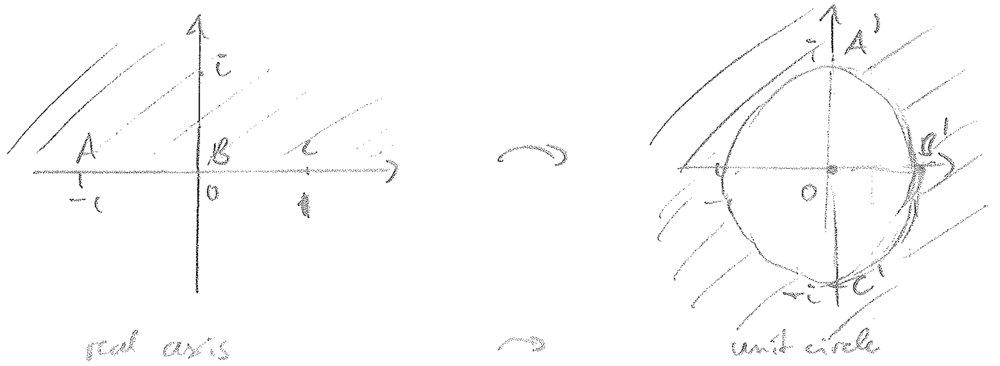
<u>Example</u> Find a Möbius transformation sending

$$-1 \mapsto i, \quad 0 \mapsto 1, \quad 1 \mapsto -i$$

$$\rightsquigarrow \quad i = \frac{-a+b}{-c+d}, \quad 1 = \frac{b}{d}, \quad -i = \frac{a+b}{c+d}$$

$$b = d, \quad \left.\begin{array}{l} -a+b+ic-id=0 \\ a+b+ic+id=0 \end{array}\right\} \quad \begin{array}{ll} \Sigma: \; 2b+2ic=0 & \rightsquigarrow c = ib = id \\ \Delta: \; 2a+2id=0 & \rightsquigarrow a = -id \end{array}$$

$$z \mapsto \frac{az+b}{cz+d} = \frac{-idz+d}{idz+d} = \frac{-iz+1}{iz+1} = \frac{z+i}{-z+i}$$



real axis $\rightsquigarrow$ unit circle

(unique circle through $i, 1, -i$)

∞	$\rightsquigarrow$	-1
i	$\rightsquigarrow$	∞
$-i$	$\rightsquigarrow$	0

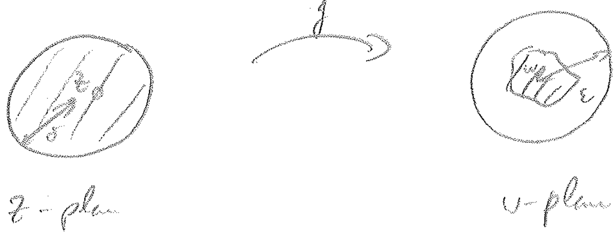
N.B.: Möbius transformations are very important in hyperbolic geometry.

Limits and Continuity

Let f be a complex function. We say that $\lim\limits_{z\to z_0} f(z) = w_0$ if "when z is close to z_0, then $f(z)$ is close to w_0". Formally

Definition $\lim\limits_{z\to z_0} f(z) = w_0$ if

$$\forall \varepsilon>0\ \exists \delta>0\ \forall\, 0<|z-z_0|<\delta : |f(z)-w_0|<\varepsilon$$



(i.e. for all ε there is always a δ such that $f(z)$ is ε-close to w_0 whenever z is δ-close to z_0)

Notes 1) We do not require that $f(z_0)$ itself be defined (that's why $0<|z-z_0|$. If $f(z_0)$ is defined, fine too)

2) When a limit exists it is unique (we cannot have $f(z)$ arbitrarily close to two different points w_0 and w_1 as $z\to z_0$

Examples 1) $f(z) = \frac{1}{2}\bar{z}$ $\quad \lim\limits_{z\to z_0} \frac{1}{2}\bar{z} = \frac{1}{2}\bar{z}_0$

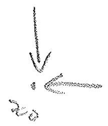
(take $\delta = 2\varepsilon$ $\forall \varepsilon>0$ $\forall\, 0<|z-z_0|<2\varepsilon$:

$$|f(z)-w_0| = \left|\tfrac{1}{2}\bar{z}-\tfrac{1}{2}\bar{z}_0\right| = \tfrac{1}{2}|\bar{z}-\bar{z}_0| = \tfrac{1}{2}|z-z_0| < \tfrac{1}{2}2\varepsilon = \varepsilon \checkmark)$$

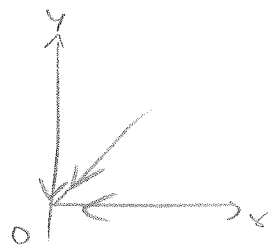
2) $f(z) = \bar{z}/z$; $\lim\limits_{z\to 0} \bar{z}/z$ does not exist!

$z = x \in \mathbb{R}$: $f(x) = 1$ $\quad z = iy \in i\mathbb{R}$: $f(iy) = -1 \neq 1$

The technique used in 2) is a common way of showing that a limit does not exist. We show that we get two different results as we approach $z \to z_0$ along two different directions



Example 3) Show that $\lim\limits_{z \to 0} \frac{z^2}{\bar{z}^2}$ does not exist



$z = x$ real : $z^2/\bar{z}^2 = x^2/x^2 = 1$

$z = iy$ imaginary : $z^2/\bar{z}^2 = -y^2/-y^2 = 1$

but $z = (1+i)t \quad (t \in \mathbb{R})$ gives

$$z^2/\bar{z}^2 = \frac{[(1+i)t]^2}{[(1-i)t]^2} = \frac{2it^2}{-2it^2} = -1 \neq 1$$

__Definition__ f is continuous at z_0 if $\lim\limits_{z \to z_0} f(z) = f(z_0)$

(i.e. if the limit exists & is equal to $f(z_0)$)

(equivalent to $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall |z - z_0| < \delta : |f(z) - f(z_0)| < \varepsilon$)

__Examples__ 1) $f(z) = az + b$ continuous at all points $z_0 \in \mathbb{C}$

2) $f(z) = \begin{cases} \bar{z}/z & z \neq 0 \\ 1 & z = 0 \end{cases}$ continuous at all points $z_0 \in \mathbb{C} \setminus \{0\}$, not continuous at $z_0 = 1$

3) $f(z) = \begin{cases} z^2/|z| & z \neq 0 \\ & z = 0 \end{cases}$ continuous at all $z_0 \in \mathbb{C}$

$\left(\lim\limits_{z \to 0} \frac{z^2}{|z|} = 0 \right)$

If we write $f(z) = u(x,y) + i v(x,y)$ then continuity of f is expressible in terms of continuity of the functions u and v,

Proposition 1.8 Let $f(z) = u(x,y) + i\, v(x,y)$. Then

$$\lim_{z \to z_0} f(z) = u_0 + i v_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \ \& \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

Note: $z = x + iy$, $z_0 = x_0 + iy_0$: $\underbrace{d\big((x,y),(x_0,y_0)\big)}_{\text{in } \mathbb{R}^2} = \underbrace{|z - z_0|}_{\text{in } \mathbb{C}}$

Proof: "$\Rightarrow$": $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall\, 0 < |z - z_0| < \delta$

$$\varepsilon > |f(z) - (u_0 + i v_0)| = |u(x,y) + i\, v(x,y) - u_0 - i v_0|$$

$$\geq \max \{ |u(x,y) - u_0|, |v(x,y) - v_0| \}$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0, \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

15.10.0

"$\Leftarrow$" $\forall \varepsilon > 0 \ \exists \delta_1 > 0 \ \forall\, 0 < d((x,y),(x_0,y_0)) < \delta_1 : |u(x,y) - u_0| < \varepsilon/2$

$\forall \varepsilon > 0 \ \exists \delta_2 > 0 \ \forall\, 0 < d((x,y),(x_0,y_0)) < \delta_2 : |v(x,y) - v_0| < \varepsilon/2$

Choose $\delta = \min(\delta_1, \delta_2)$. It follows that

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall\, 0 < \underbrace{d((x,y),(x_0,y_0))}_{=|z - z_0|} < \delta$:

$$|u(x,y) - u_0 + i\,(v(x,y) - v_0)| \leq \underbrace{|u(x,y) + u_0|}_{< \varepsilon/2} + \underbrace{|v(x,y) + v_0|}_{< \varepsilon/2} < \varepsilon$$

i.e. $|f(z) - (u_0 + i v_0)| < \varepsilon$

$\square$

Using 1.8. and standard results of real limits we have

__Proposition 1.9__ If $f(z), g(z)$ have limits w_0, w_1 as $z \to z_0$ then

(i) $f(z)+g(z)$ has limit w_0+w_1 as $z \to z_0$

(ii) $f(z)\cdot g(z)$ has limit $w_0 w_1$ as $z \to z_0$

(iii) $1/f(z)$ has limit $1/w_0$ as $z \to z_0$ (provided $w_0 \neq 0$)

And thus

__Proposition 1.10__ If $f(z), g(z)$ are continuous at $z = z_0$ then so are (1) $f(z)+g(z)$ (2) $f(z)\cdot g(z)$ (3) $1/f(z)$ (provided $f(z_0)\neq 0$)

It is obvious that a constant function $f(z)=c$ is continuous $\forall z \in \mathbb{C}$ and also that the identity $f(z)=z$ is continuous $\forall z \in \mathbb{C}$ (choose $\delta = \varepsilon$). So we deduce from (1.10) that

__Proposition 1.11__ (i) every polynomial $f(z) = a_0 + \dots + a_n z^n$ with coefficients $a_0, \dots a_n \in \mathbb{C}$ is continuous $\forall z \in \mathbb{C}$

(ii) Every rational function $f(z) = \frac{p(z)}{q(z)}$ (p, q polynomials) is continuous at all $z \in \mathbb{C}$ except possibly at the roots of $q(z)$.

__Example__

$$\lim_{z\to i} \frac{z^2+2i}{z^2+4i} = \lim_{z\to i} (z^2+2i) \lim\left(\frac{1}{z^2+4i}\right) \quad (1.9\,(ii))$$

$$= \lim_{z\to i}(z^2)+4i \Big/ \lim(z^2)+2i \quad (1.9\,(i, iii))$$

$$= \frac{i^2+4i}{i^2+2i}.$$ Thus $\frac{z^2+2i}{z^2+4i}$ is continuous at $z = i$.

It is sometimes convenient to extend the definition of limit and continuity to include the point "infinity":

<u>Definition</u>

$$\lim_{z\to\infty} f(z) = w_0 \iff \forall \varepsilon>0\ \exists R>0\ \forall |z|>R : |f(z)-w_0|<\varepsilon$$

$$\lim_{z\to z_0} f(z) = \infty \iff \forall S>0\ \exists \delta>0\ \forall |z-z_0|<\delta : |f(z)|>S$$

$$\lim_{z\to\infty} f(z) = \infty \iff \forall S>0\ \exists R>0\ \forall |z|>R : |f(z)|>S$$

Now rational functions can be regarded as continuous <u>everywhere</u> on $\mathbb{C}\cup\{\infty\}$. In practice one of the ways of dealing with "∞" in limits is to use the substitution $z=\frac{1}{\zeta}$ and let $\zeta\to 0$:

Examples: 1) $\displaystyle \lim_{z\to\infty}\frac{5z+i}{z+i} = \lim_{\zeta\to 0}\frac{5/\zeta+i}{1/\zeta+i} = \lim_{\zeta\to 0}\frac{5+i\zeta}{1+i\zeta} = 5$

2) $\displaystyle \lim_{z\to -1}\frac{iz+3}{z+1} = \infty$ since $\displaystyle \lim_{z\to -1}\frac{z+1}{iz+3} = \frac{0}{3-i} = 0$