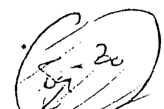
2 Differentiation

Let f be a function defined on a neighbourhood of z_0 (i.e. defined for all z with $|z-z_0|<\delta$, for some $\delta>0$)



Definition The *derivative* of f at z_0, written $f'(z_0)$ is defined to be

$$f'(z_0) = \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$$ whenever this limit exists.

If $f'(z_0)$ exists we say f is *differentiable* at z_0.

Alternative notation: write "z" for "z_0" and "$z+\Delta z$" for "z":

$$f'(z) = \lim_{\Delta z\to 0}\left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right)$$

Examples 1. $f(z)=z$

$$\lim_{\Delta z\to 0}\frac{(z+\Delta z)-z}{\Delta z} = \lim_{\Delta z\to 0}\frac{\Delta z}{\Delta z} = 1$$ so $f'(z)$ exists for all z and has value 1.

2. $f(z)=z^2$

$$\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z} = \lim_{\Delta z\to 0}\frac{2z\Delta z+(\Delta z)^2}{\Delta z}$$
$$= \lim_{\Delta z\to 0}(2z+\Delta z) = 2z$$

so $f'(z)=2z$ $(\forall z\in\mathbb{C})$

3. $f(z)=\bar z$

$$\lim_{\Delta z\to 0}\frac{\overline{z+\Delta z}-\bar z}{\Delta z} = \lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$$ which does not exist (observed earlier)

Hence $f(z)=\bar z$ is not differentiable for any $z\in\mathbb{C}$

4. $f(z)=|z|^2$

$$\lim_{\Delta z\to 0}\frac{|z+\Delta z|^2-|z|^2}{\Delta z} = \lim_{\Delta z\to 0}\frac{(z+\Delta z)(\bar z+\overline{\Delta z})-z\bar z}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\left(\bar z+\overline{\Delta z}+z\left(\frac{\overline{\Delta z}}{\Delta z}\right)\right)$$

If $z\neq 0$, $z\dfrac{\overline{\Delta z}}{\Delta z} \to \begin{cases} z & \text{as } \Delta z\to 0 \text{ along real axis} \\ -z & \text{" " " imaginary axis}\end{cases}$

so $\lim_{\Delta z\to 0}\left(\bar z+\overline{\Delta z}+z\left(\frac{\overline{\Delta z}}{\Delta z}\right)\right)$ does not exist

If $z=0$, $z\left(\frac{\overline{\Delta z}}{\Delta z}\right)=0$ so $\lim \frac{|z+\Delta z|^2-|z|^2}{\Delta z} = \bar z = 0$

Thus f is differentiable only at $z=0$, and here it has derivative 0.

Example 1 & 2 suggest differentiation is sometimes very like the real case. Examples 3 & 4 suggest it is sometimes very different. We next observe how we can build up complicated differentiable functions from elementary ones:

Proposition 2.1

(a) If f & g are differentiable at z then

(i) so is $f+g$, and $(f+g)'(z) = f'(z)+g'(z)$

(ii) so is $f.g$, and $(f.g)'(z) = f'(z)g(z) + f(z)g'(z)$

(iii) so is $\frac{f}{g}$ (if $g(z)\neq 0$), and $\left(\frac{f}{g}\right)'(z) = \dfrac{g(z)f'(z) - g'(z)f(z)}{(g(z))^2}$

(b) If g is differentiable at $f(z)$ and f is differentiable at z, then $g\circ f$ is differentiable at z and $(g\circ f)'(z) = g'(f(z)).f'(z)$.

Pf (a) follows from (1.9) (details omitted). (b) follows from the definition of derivative (just as in the real case) – details again omitted □

Example Since $f(z)=z$ is differentiable $\forall z\in\mathbb{C}$, & $f'(z)=1$, and since $f(z)=c$ is differentiable, with derivative $f'(z)=0$, it follows from (a)(i) & (a)(ii) of 2.1 that any polynomial

$$f(z) = a_0 + \dots + a_n z^n$$

is differentiable $\forall z\in\mathbb{C}$, with $f'(z) = a_1 + \dots + n a_n z^{n-1}$

It also follows from (a)(iii) of 2.1 that any rational function

$$f(z) = p(z)/q(z) \qquad (p, q \text{ polynomial})$$

is differentiable $\forall z\in\mathbb{C}$ except the roots of q, and that

$$f'(z) = \frac{q'(z)p(z) - p'(z)q(z)}{(q(z))^2}$$

End week 3

Proposition 2.2 If f is differentiable at z_0 then f is continuous at z_0.

Proof $\lim_{z\to z_0}(f(z)-f(z_0)) = \lim_{z\to z_0}\left(\dfrac{f(z)-f(z_0)}{z-z_0}\right)\lim_{z\to z_0}(z-z_0)$ by (1.9).

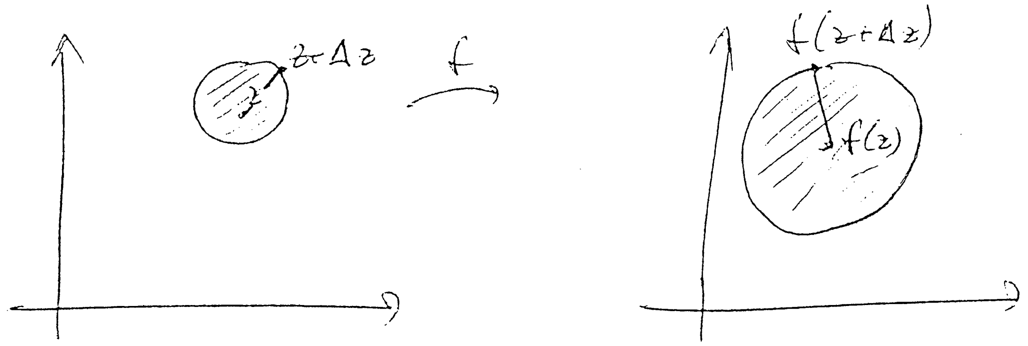
$\therefore \lim_{z\to z_0}(f(z)-f(z_0)) = f'(z_0)\times 0 = 0 \quad \therefore \lim_{z\to z_0} f(z) = f(z_0)$ □

The converse is false. For example $f(z)=\bar{z}$ is continuous everywhere yet differentiable nowhere. This leads us to the question of what a differentiable function "looks like".

Geometric interpretation of $f'(z)$

For a real function f, the derivative is the slope of the graph. What is the analogue for a complex function f?

Suppose $f'(z) \neq 0$, say $f'(z) = c = a + ib$



To a first approx, $\dfrac{f(z+\Delta z) - f(z)}{\Delta z} = c$ so $f(z+\Delta z) - f(z) = c.\Delta z$

So f multiplies a small disc by a scale factor $|c|$, and rotates it through an angle $\operatorname{Arg}(c)$. In particular at any z where f is differentiable & $f'(z) \neq 0$ the map f is conformal (angle-preserving)

WARNING When $f'(z) = 0$ the map may be angle-doubling (e.g. $z \to z^2$ at $z = 0$) or angle-trebling ($z \to z^3$ at $z = 0$) or worse.

The geometric interpretation of $f'(z)$ helps us understand the chain rule ((2.1)(b)). For if we consider the composition $g \circ f$, the first map f, to a 1st approx, multiplies by $f'(z)$ and the second, near $f(z)$ to a 1st approx multiplies by $g'(f(z))$. Hence overall $g \circ f$, near z, multiplies by $g'(f(z)) f'(z)$.

The geometric interpretation also explains why $f(z) = \bar{z}$ is not differentiable: near any z_0 this map reflects a small disc rather than turns it.

We can also now see that there are many maps $\mathbb{C} \to \mathbb{C}$ which are not differentiable since not conformal. If we write $f(z)$ as $u(x,y) + iv(x,y)$ (where $z = x + iy$) what conditions must u and v satisfy for f to be differentiable as a complex function?

Proposition 2.3 If $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z = x+iy$, then

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist at (x,y)

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at (x,y) (the Cauchy-Riemann equations)

(iii) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ at (x,y)

Proof $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

Writing $f'(z) = a+ib$ and $\Delta z = h+ik$ we get:

$$a+ib = \lim_{h,k\to 0} \frac{u(x+h,y+k) + iv(x+h,y+k) - u(x,y) - iv(x,y)}{h+ik} \quad (*)$$

We get the same value $a+ib$ however we let h,k go to 0 (by the definition of complex limits). In particular, setting $k=0$ and letting $h\to 0$ we have (from (*))

$\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial x}$ exist at (x,y), and $a+ib = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ there

But, setting $h=0$ and letting $k\to 0$ the same equation (*) tells us:

$\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y}$ exist at (x,y), and $a+ib = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ there.

Hence we deduce (i), (ii) and (iii). □

Example 1 $f(z) = z^2 = (x^2-y^2) + 2ixy$ so $u = x^2-y^2$ $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = -2y$ $\frac{\partial v}{\partial x} = 2y$ $\frac{\partial v}{\partial y} = 2x$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ all (x,y)

Moreover $f'(z) = 2z = 2x + 2iy = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Example 2 $f(z) = \bar{z}$.
Here $u = x$ $v = -y$
$\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial y} = -1$
so f is not differentiable anywhere in $\mathbb{C}$

The C-R eqns at z_0 are a necessary but not sufficient condition for differentiability at z_0 (they tell us about $\lim_{k\to 0}$ keeping $h=0$

and $\lim_{h\to 0}$ keeping $k=0$, but they don't tell us about $\Delta z \to 0$
along other directions, so they don't tell us that f is differentiable).

However there is a useful sufficient condition:

<u>Proposition 2.4</u> Let $f(z) = u(x,y) + iv(x,y)$ be defined for all z in a neighbourhood $D = \{z : |z - z_0| < r\}$ of z_0. If $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are defined everywhere in D, are continuous at $z = z_0$, and satisfy the C-R eqns at $z = z_0$, then f is differentiable at z_0 & $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ (evaluated at z_0).

<u>Proof</u> omitted (analysis). □

<u>Example 1</u> $f(z) = |z|^2 = x^2 + y^2$ $\quad u = x^2 + y^2 \quad V = 0$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0$$

These are defined, and continuous, for all $z \in \mathbb{C}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \iff x = 0 \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \iff y = 0$$

Hence f is differentiable at $z = 0$ and nowhere else. Also $f'(0) = 0$.

<u>Example 2</u> $f(z) = (x+2)^2 + i(1-y)^2$

$$u = (x+2)^2 \qquad V = (1-y)^2$$

$$\frac{\partial u}{\partial x} = 2(x+2) \quad \frac{\partial u}{\partial y} = 0 \quad \frac{\partial V}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = -2(1-y)$$

These are defined, and continuous, $\forall z \in \mathbb{C}$ so f is differentiable precisely at those points z where the C-R eqns hold i.e. where $2(x+2) = -2(1-y)$ i.e. points on the line $y = x + 3$.

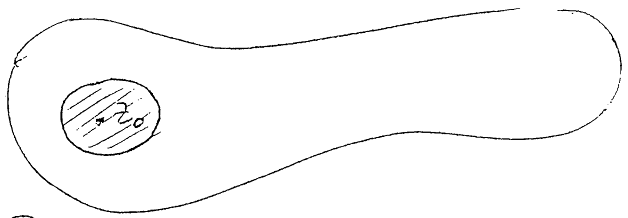
<u>Comment</u>

The C-R eqns can be thought of as expressing the <u>conformal</u> (angle-preserving) property of a differentiable map $f : \mathbb{C} \to \mathbb{C}$. This is a geometric property and does not depend on the particular coordinates used. For example (proof omitted) if we write z as $re^{i\theta}$, but continue to write $f(z) = u(r,\theta) + iv(r,\theta)$, the <u>polar-cartesian</u> form of the C-R eqns are:

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \qquad f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Open sets, domains, holomorphic functions

Definition $S \subset \mathbb{C}$ is said to be open if and only if for each $z_0 \in S$ $\exists\, r>0$ s.t. $\{z : |z-z_0|<r\} \subset S$.



Examples $\{z : |z|<1\}$ is open, but $\{z : |z| \leq 1\}$ is not

An open subset $S \subset \mathbb{C}$ is said to be connected if it is not the disjoint union of two non-empty open subsets.

Example

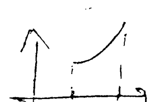
connected

disconnected

A connected open subset $\Omega \subset \mathbb{C}$ is called a domain

Proposition 2.5 If f is a complex function which is differentiable everywhere on a domain Ω and $f'(z)=0$ $\forall z \in \Omega$ then f is constant on Ω.

Proof The real version of this is proved using the mean value theorem

(

$\exists\, c \in (a,b)$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ (MVT)

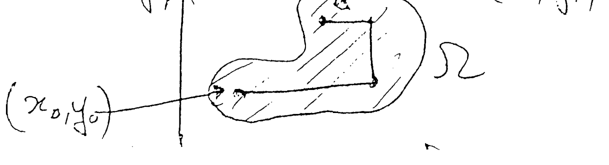
So if $f'(x)=0$ $\forall x \in (a,b)$ we deduce $f(b)=f(a)$.)

Our problem in the complex case is that there is no analogous idea of "between a & b" in the complex plane: there are many routes from a to b. So we proceed as follows:

C-R eqns $\Rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, & since $f'(z)=0$ $\forall z \in \Omega$

we have $\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = 0$ $\forall z \in \Omega$ & hence $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$ $\forall z \in \Omega$

But given $u : \mathbb{R}^2 \to \mathbb{R}$ with $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ $\forall (x,y) \in \Omega$ we can use the real MVT to prove u is constant, as follows:



Join (x_0,y_0) to (x_1,y_1) by a path which is made up of pieces parallel to the x-axis or y-axis. Use $\dfrac{\partial u}{\partial x}=0$ to show u constant

on horizontal pieces, and $\frac{\partial u}{\partial y}$ to show u is constant on vertical pieces.
Hence u is constant on Ω, and similarly so is v. Thus $f = u + iv$ is constant on Ω. $\square$

<u>Note</u> Prop. 2.5 is not true when Ω is not connected as f can take a different constant value on each "component" of Ω.

<u>Definition</u> A function f which is differentiable at every point of an open set $S \subset \mathbb{C}$ is called <u>holomorphic</u> (or <u>analytic</u>) on S. If f is differentiable at every point of $\mathbb{C}$ we say that f is <u>entire</u>.

<u>Examples</u> $f(z) = 1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$

$f(z) = \sin(z)$ is entire.