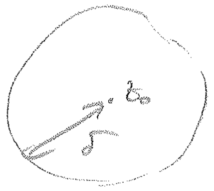
2 Differentiation

Let f be a function defined on a neighbourhood of z_0 (i.e. defined for all z with $|z-z_0|<\delta$ for some $\delta>0$)



<u>Definition</u> The <u>derivative</u> of f at z_0, written $f'(z_0)$, is defined to be
$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$
whenever the limit exists. If $f'(z_0)$ exists we say "f is <u>differentiable</u> at z_0".

<u>Alternative notation</u>:
$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

<u>Examples</u> 1) $f(z)=z$

$$\lim_{\Delta z\to 0}\frac{(z+\Delta z)-z}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta z}{\Delta z}=1$$

so $f'(z)$ exists for all $z\in\mathbb{C}$ and has value 1

2) $f(z)=z^2$

$$\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z}=\lim_{\Delta z\to 0}\frac{2z\,\Delta z+(\Delta z)^2}{\Delta z}=\lim_{\Delta z\to 0}(2z+\Delta z)=2z$$

so $f'(z)=2z\ \ \forall z\in\mathbb{C}$

3) $f(z)=\bar{z}$

$$\lim_{\Delta z\to 0}\frac{\overline{z+\Delta z}-\bar{z}}{\Delta z}=\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$$ which does <u>not</u> exist!

so $f(z)=\bar{z}$ is <u>not differentiable</u> for any $z\in\mathbb{C}$!

4) $f(z) = |z|^2$

$$\lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)\,(\overline{z}+\overline{\Delta z}) - z\overline{z})}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\overline{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z} \right)$$

$z = 0$: $\quad = \lim_{\Delta z \to 0} (\overline{\Delta z}) = 0$

$z \neq 0$: $\quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist $\leadsto$ $\lim_{\Delta z \to 0} \left(\overline{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z} \right)$ does not exist!

Thus, $f(z) = |z|^2$ is differentiable only at $z = 0$. Here, it has derivative 0.

Examples 1 & 2 suggest differentiation is sometimes very much like the real case. Examples 3 & 4 suggest it is something very different. We next observe how we can build up complicated differentiable functions from elementary ones:

<u>Proposition 2.1</u>

(a) If f & g are differentiable at z then
 (i) so is $f+g$, and $(f+g)'(z) = f'(z) + g'(z)$
 (ii) so is $f \cdot g$, and $(f \cdot g)'(z) = f'(z)\,g(z) + f(z)\,g'(z)$
 (iii) so is f/g (if $g(z) \neq 0$), and $(f/g)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$

(b) If g is differentiable at $f(z)$ and f is differentiable at z, then $g \circ f$ is differentiable at z and $(g \circ f)'(z) = g'(f(z))\,f'(z)$
(Chain rule)

<u>Proof</u> (a) follows from (1.3) (b) just as in real case, details omitted □

<u>Example</u> Since $f(z) = z$ is differentiable $\forall z \in \mathbb{C}$ with $f'(z) = 1$

and since $f(z) = c$ is differentiable $\forall z \in \mathbb{C}$ with $f'(z) = 0$,

it follows from (2.1.a.i) and (2.1.a.ii) that any

polynomial $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ is differentiable $\forall z \in \mathbb{C}$

with $f'(z) = a_1 + \ldots + n a_n z^{n-1}$

$\left(z' = 1, \; (z^n)' = (z\, z^{n-1})' = z'\, z^{n-1} + z\,(z^{n-1})' = z^{n-1} + (n-1) z^{n-1} = n z^{n-1} \right)$

It also follows from (2.1.a.iii) that any rational function

$f(z) = p(z)/q(z)$ (p, q polynomial) is differentiable $\forall z \in \mathbb{C}$

with $f'(z) = \dfrac{p'(z)\, q(z) - p(z)\, q'(z)}{(q(z))^2}$.

<u>Proposition 2.2.</u> If f is differentiable at z_0 then f is continuous at z_0

<u>Proof</u> $\lim\limits_{z \to z_0} \left(f(z) - f(z_0) \right) = \lim\limits_{z \to z_0} \left(\dfrac{f(z) - f(z_0)}{z - z_0} \right) \lim\limits_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0$

$\Rightarrow \lim\limits_{z \to z_0} f(z) = f(z_0)$ □

The converse is <u>false</u>. For example $f(z) = \bar{z}$ is continuous everywhere but nowhere differentiable.

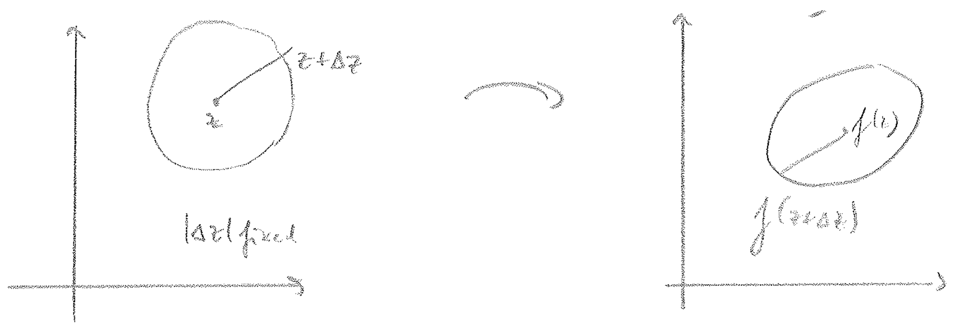
This leads us to the question of what a differentiable function "looks like".

Monday cancelled due to train (22.10.a)

Geometric interpretation of $f'(z)$

For a real function f, the derivative is the slope of the graph. What is the analogue for a complex function f?

Suppose $f'(z) \neq 0$, say $f'(z) = c = a+ib$



To a first approximation, $\frac{f(z+\Delta z) - f(z)}{\Delta z} = c$, so $f(z+\Delta z) - f(z) = c\Delta z$

So f multiplies a small distance by a scale factor $|c|$ and rotates it through an angle $Arg(c)$. In particular, at any z where f is differentiable and $f'(z) \neq 0$ the map f is angle-preserving (<u>conformal</u>)

<u>WARNING:</u> When $f'(z) = 0$ the map may be angle-doubling (e.g. $z \mapsto z^2$ at $z=0$) or worse!

The geometric interpretation of $f'(z)$ helps to understand the chain rule (2.1.5)

For if we consider the composition $g \circ f$, the first map f roughly multiplies by $f'(z)$ near z and the second roughly by $g'(f(z))$ near $f(z)$. Hence $g \circ f$ multiplies by $g'(f(z))\, f'(z)$ near z.

The geometric interpretation also explains why $f(z) = \bar{z}$ is <u>not</u> differentiable: near any z_0 this map <u>reflects</u> a small disk rather than turns it.

We can also now see that there are many maps $\mathbb{C} \to \mathbb{C}$ which are not differentiable, since not angle-preserving (conformal)

If we write $f(z)$ as $u(x,y) + i\,v(x,y)$ (where $z = x+iy$), what conditions must u and v satisfy for f to be differentiable as a complex function?

<u>Proposition 2.2</u> If $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at $z = x+iy$, then

(i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}$ all exist at (x,y)

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at (x,y)

(the <u>Cauchy-Riemann</u> differential equations)

(iii) $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ at (x,y)

<u>Proof</u>

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y} \quad (*)$$

(a) set $\Delta y = 0$:

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(b) set $\Delta x = 0$: $= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + \frac{v(x,y+\Delta y) - v(x,y)}{\Delta y} \right]$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$\rightsquigarrow$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist ; (i)

and $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$, (ii) and (iii) $\square$

<u>Example 1</u> $f(z) = z^2 = (x^2 - y^2) + i\, 2xy$

$$u(x,y) = x^2 - y^2 \quad , \quad v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

$\rightsquigarrow$ $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$ for all (x,y)

$$f'(z) = 2z = 2x + i\, 2y = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}$$

<u>Example 2</u> $f(z) = \bar{z} = x - iy$, $u = x$, $v = -y$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \quad , \quad \frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}$$

$f(z) = \bar{z}$ is nowhere differentiable.

The Cauchy-Riemann differential equations are a necessary but not sufficient condition for differentiability at z_0

(they tell us about $\lim_{\Delta x \to 0}$ keeping $\Delta y = 0$ and $\lim_{\Delta y \to 0}$ keeping $\Delta x = 0$, but they don't tell us about $\Delta z \to 0$ along arbitrary directions, so they don't tell us that f is differentiable)

However, there is a useful sufficient condition: 22.10.-6

Proposition 2.4 Let $f(z) = u(x,y) + i\,v(x,y)$ be defined for all z in a neighborhood $D = \{z : |z - z_0| < r\}$ of z_0. If $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are defined everywhere in D, are continuous at $z = z_0$, and satisfy the Cauchy-Riemann-differential equations at $z = z_0$, then f is differentiable at z_0 and $f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}$ evaluated at z_0

Proof omitted (analysis) $\square$

Example! $f(z) = |z|^2 = x^2 + y^2$

$u = x^2 + y^2$, $v = 0$; $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$

defined and continuous for all $z \in \mathbb{C}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Leftrightarrow x = 0$ $\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Leftrightarrow y = 0$

Hence f is differentiable at $z = 0$ and nowhere else. $f'(0) = 0$

<u>Example 2</u> $f(z) = (x+2)^2 + i(1-y)^2$

$u = (x+2)^2 \qquad v = (1-y)^2$

$$\frac{\partial u}{\partial x} = 2(x+2) \qquad \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = -2(1-y)$$

defined and continuous everywhere in $\mathbb{C}$. f is differentiable precisely where the Cauchy-Riemann-differential equations hold:

$$2(x+2) = -2(1-y) \quad \rightsquigarrow \quad \text{on the line } y = x+3 .$$

<u>Comment:</u> The Cauchy-Riemann differential equations can be thought of as expressing the <u>conformal</u> (angle preserving) property of a differentiable map $f: \mathbb{C} \to \mathbb{C}$. This is a geometric property and does not depend on the particular coordinates used. For example (without proof), if we write z as $re^{i\theta}$ but continue to write $f(z) = u(r,\theta) + iv(r,\theta)$, the <u>polar-cartesian</u> form of the Cauchy-Riemann differential equations is given by

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \ , \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

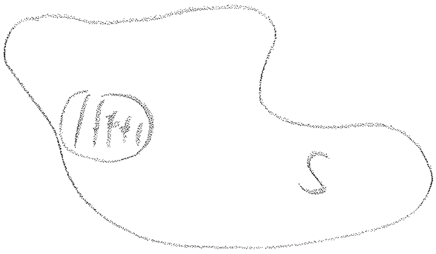
and

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Open sets, domains, holomorphic functions

Definitions $S \subset \mathbb{C}$ is said to be <u>open</u> $\Longleftrightarrow$

$$\forall z_0 \in S \ \exists r > 0 \ \{ z : |z - z_0| < r \} \subset S$$



Examples: $\{z : |z| < 1\}$ is open

$\{z : |z| \leq 1\}$ is not

$S \subset \mathbb{C}$ is said to be <u>connected</u> $\Longleftrightarrow$ S open and

S is <u>not</u> the disjoint union of two non-empty open subsets

A connected open subset $\Omega \subset \mathbb{C}$ is called a <u>domain</u>

Proposition 2.5 If f is a complex function which is differentiable everywhere on a domain Ω and $f'(z) = 0 \ \forall z \in \Omega$ then f is constant on Ω

Proof The real version of this is proved using the mean value theorem.

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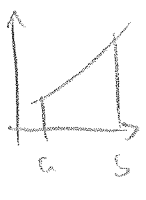


$$\exists c \in (a,b) \text{ with } f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{(MVT)}$$

$$f'(x) = 0 \ \forall x \in (a,b) \rightsquigarrow f(b) = f(a)$$

)

Our problem in the complex plane is that there is no notion of "between a and b", as there are many paths from a to b.

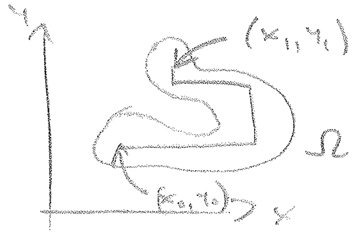
We proceed using the Cauchy-Riemann diff'eqns:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = 0 \ \forall z \in \Omega \ \leadsto \ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = 0 \ \forall z \in \Omega$$

$$\leadsto \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0 \ \forall z \in \Omega$$

But now, given $u : \mathbb{R}^2 \to \mathbb{R}$ with $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \ \forall (x,y) \in \Omega$ we can use the MVT to prove that u is constant.



Join (x_0, y_0) to (x_1, y_1) by a path which is made up of pieces parallel to the coordinate-axes.

Use $\frac{\partial u}{\partial x} = 0$ to show u constant on horizontal pieces and $\frac{\partial u}{\partial y} = 0$ to show u constant on vertical pieces. Hence u is constant on Ω, and similarly so is v. Thus $f = u + iv$ is constant on Ω. $\square$

Note Prop 7.5 is not true when Ω is not connected as f can take on different constant values on each "component" of Ω.

25.10

Definition A function which is differentiable at every point of an open set $S \subset \mathbb{C}$ is called holomorphic (or analytic) on S. If f is differentiable at every point of $\mathbb{C}$ we say that f is entire.

Examples $f(z) = 1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$

$f(z) = \sin(z)$ is entire

Holomorphic functions will play a big role during the rest of the course.