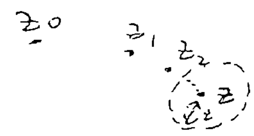
3. Power series

Basic facts about sequences and series of complex numbers

Definition The <u>sequence</u> $z_0, z_1, \ldots$ has <u>limit</u> z if for each <u>real</u> $\varepsilon > 0$ $\exists N$ such that $|z_n - z| < \varepsilon \ \forall n > N$. If the limit does not exist we say the sequence (z_n) <u>diverges</u>.



Facts

a) If the limit exists it is unique.

b) (z_n) converges to $z \iff (|z_n - z_0|)$ converges to 0
complex seq. — real seq.

Examples $z_n = \frac{i^n}{2^n}$ $\quad (z_0 = 1,\ z_1 = \frac{i}{2},\ z_2 = -\frac{1}{4}, \ldots)$ converges to 0

$z_n = (2i)^n$ $\quad (z_0 = 1,\ z_1 = 2i,\ z_2 = -4, \ldots)$ diverges

Definition The <u>series</u> $\sum_{n=0}^{\infty} z_n$ <u>converges</u> to $S (\in \mathbb{C}) \iff$ the sequence (s_n) of partial sums converges to S (where $s_n = z_0 + \ldots + z_n$). If the series does not converge, we say it <u>diverges</u>.

Facts

a) If the sum S exists it is unique.

b) Let $r_N = S - s_N$. Then $\sum_0^{\infty} z_n = S \iff \lim_{N \to \infty} |r_N| = 0$

Examples $\sum_0^{\infty} \left(\frac{i}{2}\right)^n = \frac{2}{2-i} = \frac{2}{5}(2+i)$ i.e. converges to $\frac{2}{5}(2+i)$

[Here $s_0 = 1$, $s_1 = 1 + \frac{i}{2}$, $s_2 = 1 + \frac{i}{2} - \frac{1}{4} \ldots$; proof of convergence comes from general result about geometric series.]

$\sum_0^{\infty} (2i)^n$ diverges [$s_0 = 1$, $s_1 = 1 + 2i$, $s_2 = 1 + 2i - 4, \ldots$ we shall see ~~later~~ that if $\sum z_n$ converges then $|z_n| \to 0$, so this series certainly diverges.]

Relation to real sequences & series

Proposition 3.1 (i) If $z_n = x_n + iy_n$ & $z = x + iy$ then
$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \ \& \ \lim_{n\to\infty} y_n = y$$
(ii) If $z_n = x_n + iy_n$ & $S = u + iv$ then
$$\sum_0^\infty z_n = S \iff \sum_0^\infty x_n = u \ \& \ \sum_0^\infty y_n = v$$
(iii) If $\sum_0^n z_n$ converges then $\lim_{n\to\infty} z_n = 0$ (& hence $\lim_{n\to\infty}|z_n| = 0$)

(iv) If $\sum_0^n |z_n|$ converges then $\sum_0^n z_n$ converges

Proof omitted: but worth mentioning that (i) is an exercise in δ's & ε's more or less identical to the result we proved that $\lim(f+g) = \lim f + \lim g$ for functions; (ii) is the translation of (i) into the language of series; (iii) follows from the analogous real result, via (ii). And (iv) follows from (i) since $\sum|z_n|$ converges $\Rightarrow \sum|x_n|$ converges & $\sum|y_n|$ converges $\Rightarrow \sum x_n$ converges & $\sum y_n$ converges $\Rightarrow \sum z_n$ converges. □

Omit if [illegible] time

Example: $\sum_0^\infty \left(\frac{i}{2}\right)^n = 1 + \frac{i}{2} + \frac{i^2}{4} + \dots = 1 - \frac{1}{4} + \frac{1}{16} - \dots + \frac{i}{2}\left(1 - \frac{1}{4} + \frac{1}{16} - \dots\right)$ by (ii)
$$= \frac{1}{1+\frac{1}{4}} + \frac{i}{2}\left(\frac{1}{1+\frac{1}{4}}\right) = \frac{4}{5}\left(1 + \frac{i}{2}\right) = \frac{2}{5}(2+i)$$

Note The converse to (iii) is *false*
$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \dots$$
diverges even though the terms $\frac{1}{n}$ tend to 0.

Definition If $\sum|z_n|$ converges we say that $\sum z_n$ is *absolutely convergent*

Hence (3.1)(iv) says: $\boxed{\text{abs. conv.} \Rightarrow \text{conv.}}$. The converse is false. For example $\sum(-1)^n \frac{1}{n}$ converges (alt. series test) but $\sum \frac{1}{n}$ does not.

Key example: geometric series

Prop 3.2 If $|z| < 1$, $\sum_0^\infty z^n$ converges, to $\frac{1}{1-z}$.

Proof
$$\left|\frac{1}{1-z} - S_n\right| = \left|\frac{1}{1-z} - (1 + z + \dots + z^n)\right| = \left|\frac{1}{1-z} - \frac{1-z^{n+1}}{1-z}\right| = \left|\frac{z^{n+1}}{1-z}\right|$$
But $|z| < 1 \Rightarrow |z|^{n+1} \to 0$ as $n \to \infty$

$\Rightarrow \left|\frac{1}{1-z} - S_n\right| \to 0$ as $n \to \infty$ (fixed z) $\Rightarrow S_n \to \frac{1}{1-z}$ as $n \to \infty$ □

(This proves our earlier example, where $z = i/2$)

The same proof, with $|z|$ in place of z, proves that $\sum_0^\infty z^n$ converges abs when $|z|$

<u>Complex Power Series</u>

<u>Defn</u> A <u>power series</u> in $(z-z_0)$ is a series of form $\sum_0^\infty a_n(z-z_0)^n$, where $a_n \in \mathbb{C}$. If the series converges for all $z \in U \subset \mathbb{C}$ the sum of the series defines a function $f: U \to \mathbb{C}$.

<u>Example</u> $\sum_0^\infty z^n$ is equal to the function $\frac{1}{1-z}$ on $U = \{z : |z|<1\}$ (by (3.2))

<u>Proposition 3.3</u> (Ratio Test)

For given $(z-z_0)$, if $\lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right|$ exists & is <1 then the series $\sum_0^\infty a_n(z-z_0)^n$ is abs. convergent. Conversely if the limit exists and is >1 the series is divergent.

<u>Proof omitted</u> (Essentially it is just comparing the given power series with a geometric series.) □

[If the limit does not exist, or has value 1, we have to apply other methods to find out about convergence.]

<u>Example</u> of Ratio Test The exponential function $\exp(z) = e^z$ is defined as the sum of the series $\sum_0^\infty \frac{z^n}{n!}$. Here $\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 0$ (any fixed z). Thus e^z is well-defined for all $z \in \mathbb{C}$.

<u>Note</u>
$$e^{z_1}e^{z_2} = \left(1+\frac{z_1}{1!}+\frac{z_1^2}{2!}+\dots\right)\left(1+\frac{z_2}{1!}+\frac{z_2^2}{2!}+\dots\right)$$
$$= 1 + \frac{z_1+z_2}{1!} + \frac{z_1^2+2z_1z_2+z_2^2}{2!} + \dots = e^{z_1+z_2}$$

(using the fact that we may re-arrange the order of the terms in any abs. conv. series)

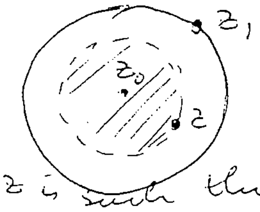
We defined $\cos z$ & $\sin z$ earlier as $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

But we could equally well define them as sums of series:
$$\cos z = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!} \qquad \sin z = \sum_0^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
which, by the ratio test, are abs. convergent for all $z \in \mathbb{C}$.

From the formula $e^{z_1+z_2} = e^{z_1}e^{z_2}$ one can deduce that the standard trigonometric formulae ($\sin 2z = 2\sin z\cos z$ etc) hold for all complex z as well as real values.

Proposition 3.4 (Radius of Convergence)
Every power series $\sum_0^\infty a_n(z-z_0)^n$ has a radius of convergence i.e. $\exists R$ (a real no. ≥ 0 or else ∞) such that if $|z-z_0|<R$ then $\sum_0^\infty a_n(z-z_0)^n$ converges absolutely, and if $|z-z_0|>R$ then $\sum_0^\infty a_n(z-z_0)^n$ diverges.

Proof It will suffice to prove that if $\sum_0^\infty a_n(z_1-z_0)^n$ converges, then $\sum_0^\infty a_n(z-z_0)^n$ converges abs. $\forall z$ with $|z-z_0|<|z_1-z_0|$.



So, suppose $\sum_0^\infty a_n(z_1-z_0)^n$ converges & z is such that $|z-z_0|<|z_1-z_0|$.
Then $a_n(z_1-z_0)^n \to 0$ as $n\to\infty$, so $\exists M$ (real no.) such that $a_n(z_1-z_0)^n<M$ $\forall n$.
But now $|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \cdot \left|\frac{z-z_0}{z_1-z_0}\right|^n < Mb^n$, where $b = \left|\frac{z-z_0}{z_1-z_0}\right| < 1$. So $\sum_0^\infty |a_n(z-z_0)^n|$ converges, by the comparison test $\square$

Note R exists, whether or not we can find it by the Ratio Test ...
Write $D=\{z : |z-z_0|<R\}$ for the disc of convergence of the series $\sum_0^\infty a_n(z-z_0)^n$, and write $f(z)$ for the sum of the series.

Proposition 3.5 (i) $f(z)=\sum_0^\infty a_n(z-z_0)^n$ is holomorphic on D
(ii) for all $z\in D$ the series $\sum_{n=0}^\infty n a_n(z-z_0)^{n-1}$ is abs. convergent and has sum $f'(z)$, the derivative of f at z.

Proof This can be proved using ε's and δ's, or it can also be proved as a consequence of our results later in complex integration theory. $\square$ (We shall do the latter proof if we have time.)

Note that it follows from Prop 3.5 that $f(z) = \sum_0^\infty a_n(z-z_0)^n$ is differentiable <u>arbitrarily many times</u> at any z in the disc of convergence of the series.

(PF Cauchy 1831)

(Brook Taylor 1715 real Taylor series)

A much more surprising theorem is:

<u>Thm 3.6</u> (Taylor Series) Let f be holo. on the disc $D = \{z : |z-z_0| < R\}$. Then f is differentiable arbitrarily many times at z_0, and $\forall z \in D$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{n}(z_0)(z-z_0)^n$$

i.e. the series on the right, called <u>the Taylor series for f at z_0</u>, converges to $f(z)$. Moreover this is the unique representation of f as a power series $\sum_0^\infty a_n(z-z_0)^n$ on D

<u>Proof</u> later in the course, using complex integration theory $\square$

(i.e. once diff'ble fn)

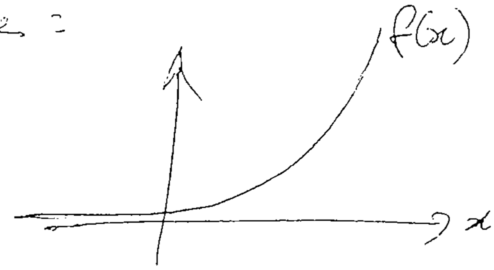
Notice that (3.6) says that any holo. fn can be written as a power series, and applying (3.5) we can deduce that

$$f'(z) = \sum_1^\infty \frac{1}{(n-1)!} f^{(n)}(z_0)(z-z_0)^{n-1}, \quad f''(z) = \sum_2^\infty \frac{1}{(n-2)!} f^{(n)}(z_0)(z-z_0)^{n-2},$$

etc, $\forall z \in D$. This is very different from the real case where a function can be differentiable once without being differentiable twice.

Indeed a real everywhere differentiable function need not be equal to the sum of its Taylor series:

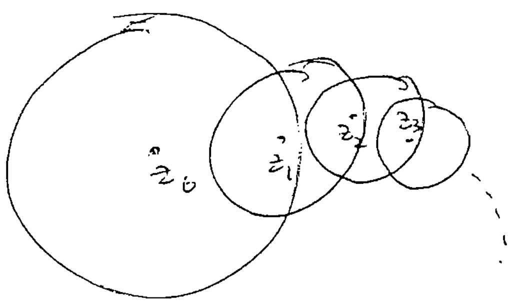
<u>Example</u> $f(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases}$



This has $f^{(n)}(0) = 0$ $\forall n$ so its Taylor series is $0 + 0x + 0x^2 + \dots$, but $f(x)$ is <u>not</u> the zero function

The complex version of Taylor's Thm (3.6) tells us that once we know $f(z_0), f'(z_0), \ldots, f^{(n)}(z_0), \ldots$ we know the value of $f(z)$ <u>everywhere</u> in the disc of convergence of the Taylor series $\sum_0^\infty \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$. This leads to the idea of <u>analytic continuation</u> :–

not discussed in lectures, but it is not forbidden to read this!



Suppose we know $f(z_0), f'(z_0), \ldots, f^{(n)}(z_0) \ldots$ Work out the Taylor series centred on z_0. This has a radius of convergence. Choose z_1 in the disc of convergence, near the edge. Work out the Taylor series centred at z_1. This new power series has a radius of convergence ... etc.

Hence creep round $\mathbb{C}$ extending the region in which we know f. And we started only with information at the point z_0 !

<u>Examples of Taylor Series</u> [If time is restricted, the important example to give is 3)]

i) $e^z = \sum_0^\infty \frac{z^n}{n!}$ (by defn.)

This series converges abs $\forall z \in \mathbb{C}$ (ratio test) so $R = \infty$.
Hence e^z is differentiable $\forall z \in \mathbb{C}$ & has derivative $\sum_0^\infty \frac{nz^{n-1}}{n!}$ (3.5).
But this is just e^z again.

$$\therefore \frac{d^{(n)}(e^z)}{dz^{(n)}} = e^z, \text{ which evaluated at } z=0 \text{ is } 1.$$

Hence the Taylor series for e^z about $z=0$ is $\sum_0^\infty \frac{z^n}{n!}$ (which we could have deduced from the uniqueness statement in (3.6))

2) $\dfrac{1}{2z-3}$ about $z=2$?

Method 1 Formula: with $f(z) = \dfrac{1}{2z-3}$ compute $f^{(n)}(z)$ & ~~set~~ hence Taylor series

$$\sum_0^\infty \frac{f^{(n)}(2)}{n!}(z-2)^n$$

Method 2 $f(z) = \dfrac{1}{2z-3} = \dfrac{1}{2}\left(\dfrac{1}{z-3/2}\right)$

$$= \frac{1}{2}\cdot\frac{1}{(z-2)+\frac{1}{2}} = \frac{1}{1+2(z-2)}$$

$$= 1 - 2(z-2) + 4(z-2)^2 - 8(z-2)^3 + \dots$$

Abs convergent for $|2(z-2)| < 1$

i.e. $|z-2| < \frac{1}{2}$

$\dfrac{1}{2z-3}$ undefined at $z = 3/2$

Laurent Series

Taylor series exist for functions which are holomorphic on discs. Holomorphic functions on certain other type of region also have series representations.

Examples $\dfrac{1}{1-z}$. We have a power series in z for $|z|<1$. What can we do for $|z|>1$? Then $\left|\frac{1}{z}\right|<1$ and we can use a series in $\frac{1}{z}$:-

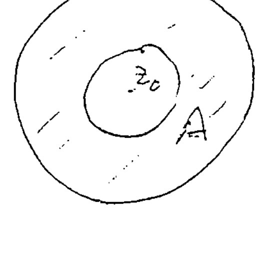
$$\frac{1}{1-z} = \frac{1}{z}\cdot\frac{1}{\frac{1}{z}-1} = \left(-\frac{1}{z}\right)\cdot\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}\cdots$$

$$= \sum_{n=1}^\infty (-z^{-n})$$

Abs convergent $\forall z$ with $|z|>1$ (divergent $\forall z$ with $|z|<1$)

Definition

Let A be an annulus, $A = \{z : R_1 < |z - z_0| < R_2\}$, centred at z_0 (where $0 \le R_1 < R_2 \le \infty$).



A series $\sum_0^\infty a_n (z-z_0)^n + \sum_1^\infty b_n (z-z_0)^{-n}$ converging to $f(z)$ at every $z \in A$, is called a Laurent Series for f on A. [Laurent introduced them in 1843]

Note: Sometimes write $\sum_{-\infty}^{+\infty} a_n (z-z_0)^n$ but then have to remember this is really the sum of two series.

Theorem 3.7 (Laurent) Let f be holomorphic on the annulus A, centred at z_0. Then $f(z)$ has a unique Laurent Series expansion $\sum_0^\infty a_n (z-z_0)^n + \sum_1^\infty b_n (z-z_0)^{-n}$ converging absolutely to $f(z)$ at every point of A.

Proof later, using complex integration. There are formulae for the a_n, involving integrals, but they are not very practical for computing the a_n. □

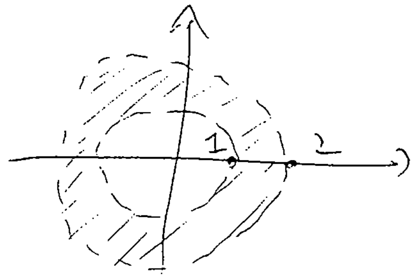
Examples 1) $\frac{1}{1-z}$ has Laurent series $\sum_1^\infty -(z^{-n})$ on $A = \{z : 1 < |z| < \infty\}$

and " " $\sum_0^\infty z^n$ on $A = \{z : 0 < |z| < 1\}$

(Note that Laurent Series = Taylor series when the domain is a disc.)

2) $e^{\frac{1}{z}} = \sum_0^\infty \frac{z^{-n}}{n!}$ (by defn.) on $A = \{z : 0 < |z| < \infty\}$ so this is its Laurent series there (by uniqueness)

3) $f(z) = \dfrac{1}{(z-1)(z-2)}$

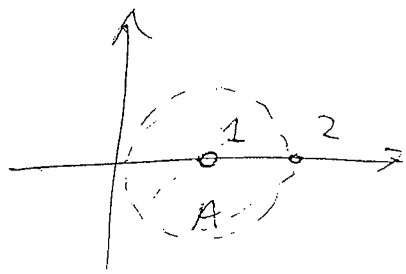


Laurent series on
$A = \{z : 1 < |z| < 2\}$?

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{-1}{2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}$$
$$= -\frac{1}{2}\sum_0^\infty \left(\frac{z}{2}\right)^n - \sum_1^\infty z^{-n}$$

4) $f(z) = \dfrac{1}{(z-1)(z-2)}$

Laurent series on
$A = \{z : 0 < |z-1| < 1\}$?

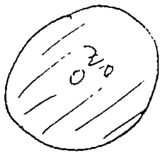


$$f(z) = \frac{1}{1-z}\cdot\frac{1}{2-z}$$
$$= \frac{1}{1-z}\cdot\frac{1}{1-(z-1)} = \frac{1}{1-z} \times \sum_0^\infty (z-1)^n$$
$$= -(z-1)^{-1} + \sum_0^\infty -(z-1)^n$$

In this last example we are considering a Laurent series expansion on a <u>punctured disc</u> $D' = \{z : 0 < |z - z_0| < r\}$. This is where most of our Laurent series will be from now on.

<u>Singularities, poles, residues</u>

<u>Definition</u> If f is holomorphic on a punctured disc $D' = \{z : 0 < |z - z_0| < r\}$ but it is not holomorphic, or maybe not defined, at z_0, then z_0 is called an <u>isolated singularity</u> of f.

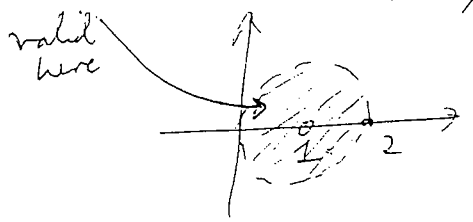


Disc with single point z_0 removed

By Laurent's Theorem, such an f has a Laurent series
$$f(z)=\sum_0^\infty a_n(z-z_0)^n+\sum_1^\infty b_n(z-z_0)^{-n} \text{ valid on } D.$$

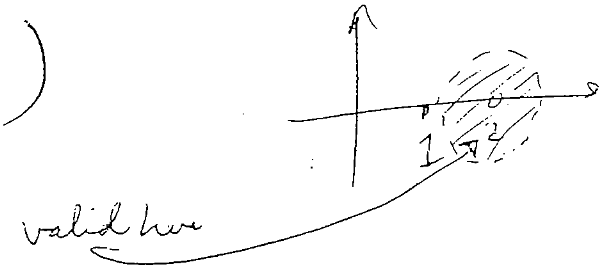
<u>Example</u> $f(z)=\dfrac{1}{(z-1)(z-2)}$ has isolated singularities at $z=1,2$

<u>Laurent series about $z=1$:</u> $\quad -(z-1)^{-1}+\sum_0^\infty -(z-1)^n \quad$ (just done)



<u>Laurent series about $z=2$?</u>

$$f(z)=\frac{1}{z-2}\cdot\frac{1}{1+(z-2)}$$
$$=\frac{1}{z-2}\cdot\left(1-(z-2)+(z-2)^2-\dots\right)$$
$$=(z-2)^{-1}+\sum_0^\infty(-1)^n(z-2)^n$$



<u>Terminology</u>

If $f(z)$ has an isolated singularity at $z=z_0$ and Laurent series $\sum_0^\infty a_n(z-z_0)^n+\sum_1^\infty b_n(z-z_0)^{-n}$ then

(i) the part $\sum_1^\infty b_n(z-z_0)^{-n}$ is called the <u>principal part</u> of the Laurent series (the rest tend to a_0 as $z\to z_0$);

(ii) the coefficient b_1 (of $(z-z_0)^{-1}$) is called the <u>residue</u> of the singularity at z_0 (it has a special role in integration theory).

There are 3 types of isolated singularity:

(a) If the principal part has a finite number (>0) of non-zero terms, i.e. if it has form $\sum_{n=1}^m b_n(z-z_0)^{-n}$ (with $b_m\neq 0$) then z_0 is called a <u>pole of order m</u>.

A pole of order 1 is called a simple pole

(b) If $b_n \neq 0$ for infinitely many n then z_0 is called an essential singularity.

(c) If $b_n = 0$ $\forall n$ (i.e. the principal part is 0) we say that z_0 is a removable singularity. We can then define $f(z_0)$ to be a_0 and $\sum_0^\infty a_n(z-z_0)^n$ will be a Taylor series, valid on the whole disc $D = \{z : 0 \leq |z-z_0| < r\}$

The type of an isolated singularity can be recognised by the behaviour of f near to it:

(a) If z_0 is a pole then $\lim_{z \to z_0} f(z) = \infty$

[Obvious: if z_0 is a pole of order m then

$$f(z) = b_m(z-z_0)^{-m} + \dots + b_1(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \dots$$

$\therefore (z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \dots$

$\therefore (z-z_0)^m f(z)$ tends to b_m as z tends to z_0

But $(z-z_0)^m$ tends to 0 so $f(z)$ tends to ∞.]

(b) If z_0 is an essential singularity then $\lim_{z\to z_0} f(z)$ does not exist.

[Not obvious. But more is true. Picard (1856-1941) proved that in any nbhd of an isolated essential singularity z_0, f takes every value in $\mathbb{C}$, with at most one exception.

(c) If z_0 is removable, $\lim_{z\to z_0} f(z)$ is finite $(= a_0)$. [Obvious]

Examples

1) $e^{1/z}$: isolated sing. at $z=0$; principal part has infinitely many terms so $z=0$ is an essential singularity. Coeff of z^{-1} is 1 so $\mathrm{Res}_{z=0}\, e^{1/z} = 1$.

[N.B. In every nbhd of $z=0$, however small, $e^{1/z}$ takes every value in $\mathbb{C}$ except 0.]

2) $f(z) = \dfrac{1}{(z-1)(z-2)}$ has an isolated sing. at $z=1$

Laurent series: $-(z-1)^{-1} - 1 - (z-1) - (z-1)^2 - \dots$

So $z=1$ is a simple pole, residue -1.

(the singularity at $z=2$ is also a simple pole, residue $+1$)

3) $f(z) = \dfrac{e^z - 1}{z}$ has an isolated sing. at $z=0$ (since undefined there)

Laurent series $\frac{1}{z}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots \; -1\right) = \frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \dots$

Hence the singularity at $z=0$ is removable. (Define $f(0)=1$)

Proposition 3.8 If $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ where ϕ is hol. at z_0, with $\phi(z_0) \neq 0$, and where $m>0$, then f has a pole of order m at z_0, residue $\dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ ($\Leftarrow$ by Laurent series)

Proof

$\phi(z)$ has Taylor series $\phi(z_0) + \dfrac{\phi'(z_0)}{1!}(z-z_0) + \dots$ in some nbhd of z_0.

Hence $f(z) = \dfrac{\phi(z_0)}{(z-z_0)^m} + \dfrac{\phi'(z_0)}{1!}\dfrac{1}{(z-z_0)^{m-1}} + \dots$, a Laurent series with finitely many terms in the principal part. The residue (coeff of $(z-z_0)^{-1}$) is $\dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$. $\square$

Examples 1) $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{\phi(z)}{z-2}$ where $\phi(z) = \dfrac{1}{z-1}$

$\phi(2) = 1 \neq 0$ so f has a simple pole at $z=2$, residue $\dfrac{\phi(2)}{0!} = \phi(2) = 1$.

2) $f(z) = \dfrac{z+1}{z^2+9}$ Singularities $z = \pm 3i$.

$z=3i$: $f(z) = \dfrac{\phi(z)}{z-3i}$ where $\phi(z) = \dfrac{z+1}{z+3i}$ so $\phi(3i) = \dfrac{3-i}{6} \neq 0$

Hence f has simple pole at $3i$, residue $(3-i)/6$.

(similarly $z=-3i$ is a simple pole, residue $\frac{3+i}{6}$)

3) $f(z) = \dfrac{z^3+2z}{(z-i)^3}$ Singularity $z=i$.

$$f(z) = \frac{\phi(z)}{(z-i)^3} \qquad \phi(z) = z^3+2z$$
$$\phi(i) \neq 0$$

so f has pole order 3 at $z=i$, $\text{res} = \dfrac{\phi^{(2)}(i)}{2!} = \dfrac{6i}{2} = 3i$

4) $f(z) = \dfrac{\sin z}{z^4}$ has a pole of order 3 (not 4) at $z=0$

since $f(z) = \dfrac{1}{z^4}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} \cdots\right) = \dfrac{1}{z^3} - \dfrac{1}{3!\,z} + \dfrac{z}{5!} - \dfrac{z^3}{7!} + \cdots$

$$\operatorname*{Res}_{z=0} f = -\frac{1}{6}$$

[**Note** The importance of residues will become much clearer later, when we do integration. But observe that the z^{-1} term is the tricky one to integrate.]

Zeros & Poles

Definition If $f(z) = (z-z_0)^m g(z)$ where $m>0$ & $g(z_0)\neq 0$ we say f has a zero of order m at z_0.

Example $z(e^z-1) = z^2\left(1 + \dfrac{z}{2!} + \dfrac{z^2}{3!} + \cdots\right)$ has a zero of order 2 at $z=0$

Prop 3.9

Observe that f has a zero of order m at z_0 $\iff$ $f(z_0), \dots, f^{(m-1)}(z_0)$ are all 0 but $f^{(m)}(z_0) \neq 0$.

If p, q are holo at z_0 & $p(z_0)\neq 0$ then $f = p/q$ has a pole of order m at $z_0 \iff q$ has a zero order m at z_0

Pf $\Rightarrow$: f pole order m at z_0 $\Rightarrow f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ ϕ holo at z_0 & $\phi(z_0)\neq 0$

$\Rightarrow q(z) = (z-z_0)^m \dfrac{p(z)}{\phi(z)}$ with $\dfrac{p(z_0)}{\phi(z_0)} \neq 0$

$\Rightarrow q$ has a zero order m at z_0.

$\Leftarrow$: if q has a zero order m at z_0

$q(z) = (z-z_0)^m g(z)$ with $g(z_0) \neq 0$

so $f(z) = \dfrac{p(z)}{q(z)} = \dfrac{1}{(z-z_0)^m} \dfrac{p(z)}{g(z)}$ with $\dfrac{p(z_0)}{g(z_0)} \neq 0$

So f has a pole of order m at z_0. $\square$

Cor 3.10 If $p(z_0) \neq 0$ & q has a simple zero at z_0 then p/q has a simple pole at z_0 & $\operatorname{Res}_{z_0} p/q = \dfrac{p(z_0)}{q'(z_0)}$.

Pf By 3.9 with $m=1$ p/q has a simple pole at z_0. To find the residue observe that $q(z) = \underbrace{q(z_0)}_{=0} + (z-z_0)q'(z_0) + \dots$ (Taylor series)

So $\dfrac{p(z)}{q(z)} = \dfrac{1}{z-z_0} \cdot \dfrac{p(z_0)}{q'(z_0) + \frac{q''(z_0)}{2!}(z-z_0) + \dots}$ $\square$

Examples

1) $\dfrac{1}{z(e^z-1)}$: singularities at $z = 2n\pi i$

$n \neq 0$: $p(z) = 1$ $q(z) = z(e^z - 1)$

$z = 2n\pi i$ $(n \neq 0)$ is a simple zero of q.

$q'(z) = ze^z + e^z - 1$ so $q'(2n\pi i) = 2n\pi i$

$\therefore \dfrac{1}{z(e^z-1)}$ has a simple pole at $z = 2n\pi i$,

residue $\dfrac{1}{2n\pi i}$

$n=0$: $z=0$ is a double zero since $\dfrac{1}{z(e^z-1)} = \dfrac{1}{z^2\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)}$

Residue here = coeff of $\frac{1}{z}$ in $\dfrac{1}{z^2}\left(1+\dfrac{z}{2!}+\dfrac{z^2}{3!}+\dots\right)^{-1}$

i.e. $-\dfrac{1}{2}$ (see Prop. 3.8.)