3. Power Series

Basic facts about sequences and series of complex numbers

Definition The sequence $z_0, z_1, z_2, \ldots$ has limit z $\Leftrightarrow$

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N}_0 \; \forall n > N : |z_n - z| < \varepsilon$$

If the limit does not exist we say the sequence (z_n) diverges

Facts a) If the limit exists it is unique

b) (z_n) converges to z $\Leftrightarrow$ $(|z_n - z_0|)$ converges to zero
$\uparrow$ complex $\qquad$ $\uparrow$ real

Examples: $z_n = \dfrac{i^n}{2^n}$ $\quad (z_0 = 1, \; z_1 = \frac{i}{2}, \; z_2 = -\frac{1}{4}, \ldots)$ converges to 0

$z_n = (2i)^n$ $\quad (z_0 = 1, \; z_1 = 2i, \; z_2 = -4, \ldots)$ diverges

Definition The series $\sum_{n=0}^{\infty} z_n$ converges to $s \; (\in \mathbb{C})$ $\Leftrightarrow$

the sequence (S_n) of partial sums $S_n = \sum_{k=0}^{n} z_k$ converges to S

If the series does not converge, we say it diverges

Facts a) If the sum S exists it is unique

b) Let $r_N = S - S_N$. Then $\sum_{n=0}^{\infty} z_n = S \Leftrightarrow \lim_{N \to \infty} |r_N| = 0$

Examples $\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = \dfrac{1}{1 - \frac{i}{2}} = \dfrac{2}{2-i} = \frac{2}{5}(2+i)$

$S_0 = 1, \; S_1 = 1 + \frac{i}{2}, \; S_2 = 1 + \frac{i}{2} - \frac{1}{4}, \ldots$ $\qquad$ proof from general result on geometric series

$\sum_{n=0}^{\infty} (2i)^n = 1 + 2i - 4 - 8i + \ldots$ diverges

<u>Relation to real sequences and series</u>

<u>Proposition 3.1</u> (i) If $z_n = x_n + iy_n$ & $z = x + iy$ then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \ \&\ \lim_{n\to\infty} y_n = y$$

(ii) If $z_n = x_n + iy_n$ & $s = u + iv$ then

$$\sum_{n=0}^{\infty} z_n = s \iff \sum_{n=0}^{\infty} x_n = u \ \&\ \sum_{n=0}^{\infty} y_n = v$$

(iii) If $\sum_{n=0}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n = 0$

(iv) If $\sum_{n=0}^{\infty} |z_n|$ converges then $\sum_{n=0}^{\infty} z_n$ converges

<u>Proof:</u> (i) like $\lim(f+g) = \lim f + \lim g$

(ii) sequences $\to$ series

(iii) follows from "real" result (iv) $\sum_{n=0}^{\infty} |z_n|$ converges $\Rightarrow \sum_{n=0}^{\infty} |x_n|, \sum |y_n|$ conv

$\Rightarrow \sum_{n=0}^{\infty} x_n, \sum_{n=0}^{\infty} y_n$ conv. Use (ii) $\square$

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Example: $$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = 1 + \frac{i}{2} - \frac{1}{4} - \frac{i}{8} + \dots = \left(1 - \frac{1}{4} + \frac{1}{16} - \dots\right) + i\left(\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \dots\right)$$

$$= \frac{1}{1+\frac{1}{4}} + i\,\frac{\frac{1}{2}}{1+\frac{1}{4}} = \frac{4}{5} + i\,\frac{2}{5} = \frac{2}{5}(2+i)$$

Reminder: Coursework mark based on best 6 weekly ~~exercises~~ & also 15% max. mark ~~in first sheet~~!

<u>Note:</u> The converse to (iii) is <u>false</u>!

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \dots$$ diverges even though $\frac{1}{n} \to 0$

<u>Definition</u> If $\sum_{n=0}^{\infty} |z_n|$ converges we say that $\sum_{n=0}^{\infty} z_n$ is <u>absolutely convergent</u>

Hence (3.1.iv) says: absolute convergence implies convergence. The converse is false

For example $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges $(= \log 2)$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not.

Key example: geometric series

Prop. 3.2 If $|z|<1$, $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$

Proof $S_n = \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$ $\qquad S - S_n = \frac{1}{1-z} - \frac{1-z^{n+1}}{1-z} = \frac{z^{n+1}}{1-z}$

$$|S - S_n| = \left|\frac{z^{n+1}}{1-z}\right| \to 0 \text{ as } n\to\infty \quad (|z|^n \to 0 \text{ for } |z|<1)$$ □

(This proves our earlier example with $z = \frac{1}{2}$)

The same proof, with $|z|$ in place of z, proves that $\sum_{n=0}^{\infty} z^n$ converges absolutely for $|z|<1$

Complex Power Series

Definition: A power series in $(z-z_0)$ is a series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n \in \mathbb{C}$. If the series converges for all $z \in U \subset \mathbb{C}$, the sum of the series defines a function $f: U \to \mathbb{C}$

Example: $\sum_{n=0}^{\infty} z^n$ is equal to the function $\frac{1}{1-z}$ on $U = \{z : |z|<1\}$

Proposition 3.3 (Ratio Test)

For given $(z-z_0)$, if $\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n (z-z_0)^n} \right|$ exists and is <1 then the series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is absolutely convergent. Conversely, if the limit exists and is >1 the series is divergent.

Proof: Ratio Test for real sequences: $c_n = |a_n (z-z_0)^n|$, $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$ implies convergence for $\sum_{n=0}^{\infty} c_n$. This implies absolute convergence in (3.3) □

Note If the limit does not exist, or has value 1, we have to apply other methods to find out about convergence.

Example (of Ratio Test): The exponential function $\exp(z) = e^z$ is defined as the sum of the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Here $a_n = \frac{z^n}{n!}$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$= \lim_{n\to\infty} \frac{z}{n+1} = 0$ (for all $z \in \mathbb{C}$ fixed) Thus e^z is well-defined $\forall z \in \mathbb{C}$

Note: $$e^{z_1} e^{z_2} = \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{\ell=0}^{\infty} \frac{z_2^\ell}{\ell!} = \sum_{n=0}^{\infty} \sum_{k+\ell=n} \frac{z_1^k z_2^\ell}{k!\,\ell!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (z_1+z_2)^n = e^{z_1+z_2}$$

rearrangement of absolutely conv series is allowed. (2.10.5)

We defined $\cos z$ and $\sin z$ earlier as $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

But we could well define them as series $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$, $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

By the ratio test, these are abs. convergent for all $z \in \mathbb{C}$.

From $e^{z_1+z_2} = e^{z_1} e^{z_2}$ one can deduce that the standard trigonometric formulae ($\sin 2z = 2\sin z \cos z$) hold for all complex z as well as real values

Example: $$\cos nz + i \sin nz = e^{inz} = (e^{iz})^n = (\cos z + i\sin z)^n$$

e.g. $n=3$: $\cos 3z + i\sin 3z = \cos^3 z + 3i\cos^2 z \sin z - 3\cos z \sin^2 z - i\sin^3 z$

$\rightsquigarrow$ $\cos 3z = \cos^3 z - 3\cos z \sin^2 z = 4\cos^3 z - 3\cos z$

$\sin 3z = 3\cos^2 z \sin z - \sin^3 z = 3\sin z - 4\sin^3 z$

(first, use z real, then extend to $z \in \mathbb{C}$)

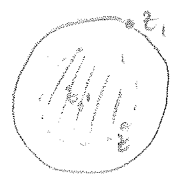
Proposition 3.4 (Radius of convergence)

Every power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has a radius of convergence i.e.

$\exists R \geq 0$ (may be "∞") such that if $|z-z_0| < R$ then

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely, and if $|z-z_0| > R$ then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ diverges

Proof It will suffice to prove that if $\sum_{n=0}^{\infty} a_n (z_1-z_0)^n$ converges then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges absolutely for all z with $|z-z_0| < |z_1-z_0|$



Suppose $\sum_{n=0}^{\infty} a_n (z_1-z_0)^n$ converges $\rightsquigarrow$ $a_n(z_1-z_0)^n \to 0$ as $n\to\infty$

$\rightsquigarrow |a_n(z-z_1)^n| < M$ bounded by M for all n

Suppose next $|z-z_0| < |z_1-z_0|$. Then $a_n(z-z_0)^n = a_n(z_1-z_0)^n \left(\frac{z-z_0}{z_1-z_0}\right)^n$

and $|a_n(z_1-z_0)^n| < M \left|\frac{z-z_0}{z_1-z_0}\right|^n = M b^n$ where $b = \left|\frac{z-z_0}{z_1-z_0}\right| < 1$.

Thus, $\sum_{n=0}^{\infty} |a_n(z_1-z_0)^n|$ converges by the comparison test.

Note: R exists, whether or not we can find it by the ratio test. Write

$D = \{z : |z-z_0| < R\}$ for the disk of (absolute) convergence of the series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, and write $f(z)$ for the sum of the series.

Proposition 3.5 (i) $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is holomorphic on D

(ii) for all $z \in D$ the series $\sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$ is absolutely convergent and has sum $f'(z)$, the derivative of f at z

Proof: This can be proved using ε's and δ's, or as a consequence of later results in complex integration theory (shall do so later) $\square$

Note that it follows from Prop 3.5 that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is differentiable arbitrarily many times at any z in the disk of convergence of the series.

A much more surprising fact is:

Theorem 3.6 (Taylor Series) Cauchy 1831, "real Taylor series": Brook Taylor, 1715

Let f be holomorphic on the disk $D = \{z : |z-z_0| < R\}$. Then f is differentiable arbitrarily many times at z_0, and $\forall z \in D$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n$$

i.e. the series on the right, called the Taylor series for f at z_0, converges to $f(z)$. Moreover this is the unique representation of f as a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ on D

Proof. Later using complex integration theory $\square$

Note that (3.6) implies that any holomorphic function (i.e. defined as differentiable once) can be written as a power series and by (3.5) is infinitely often differentiable:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n, \quad f'(z) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} f^{(n)}(z_0)(z-z_0)^{n-1}, \quad f''(z) = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} f^{(n)}(z_0)(z-z_0)^{n-2}$$

etc.

This is very different from the real case where a function can be differentiable only once.

A real, everywhere differentiable function need not be equal to its Taylor series:

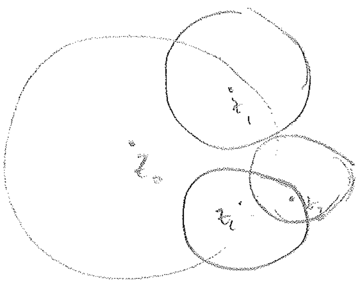
$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases} \qquad f^{(n)}(0) = 0 \quad \forall n$$

Taylor series is $\quad 0 + 0x + 0x^2 + \ldots = 0 \neq f(x)$

The complex version of Taylor's theorem (3.6) implies also that knowledge of $f^{(n)}(z_0)$ $\forall n \in \mathbb{N}_0$ implies knowledge of $f(z)$ in the whole disk of convergence of the Taylor series: $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$.

— (1.11) —

This leads to the idea of analytic continuation:



Suppose we know $f^{(n)}(z_0)$ $\forall n \in \mathbb{N}_0$
$\rightsquigarrow$ know $f(z)$ inside disk centred at z_0

Pick new value z_1 "close to border" and compute $f^{(n)}(z_1)$ $\rightsquigarrow$ know $f(z)$ inside disk centred at z_1

Repeating this process extends the Domain in which we know f. And we started only with information of f at the point z_0!

Examples of Taylor Series

1) $\quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad$ (by definition)

converges absolutely $\forall z \in \mathbb{C}$ (ratio test) $\Rightarrow R = \infty$

Hence e^z is differentiable $\forall z \in \mathbb{C}$ and has derivative $\sum_{n=0}^{\infty} \frac{n z^{n-1}}{n!}$

which is e^z.

Repeat: $\quad \left(\frac{d}{dz}\right)^n e^z = e^z, \quad \left(\frac{d}{dz}\right)^n e^z \Big|_{z=0} = 1$. Taylor series at $z_0=0$ is $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

e^z at $z_0 = 1$:

i) $\left(\frac{d}{dz}\right)^n e^z \Big|_{z=z_0} = e^z \Big|_{z=z_0} = e^{z_0} \rightsquigarrow f(z) = \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!} (z-z_0)^n$

ii) $e^z = e^{z_0} \underbrace{e^{z-z_0}}_{e^t,\ t=z-z_0} = e^{z_0} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!} (z-z_0)^n$

2) $\frac{1}{z}$ at $z_0 = 1$ $\qquad \left(\frac{1}{z}\right)' = -\frac{1}{z^2},\ \left(\frac{1}{z}\right)'' = \frac{2}{z^3}\ \left(\frac{1}{z}\right)''' = -\frac{6}{z^4}, \ldots$

i) $\left(\frac{d}{dz}\right)^n \left(\frac{1}{z}\right) \Big|_{z=1} = \frac{(-1)^n n!}{z^{n+1}} \Big|_{z=1} = (-1)^n n! \rightsquigarrow f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{\cancel{n!}} (z-1)^n$

$= \sum_{n=0}^{\infty} (1-z)^n \ ; \ |z-1| < 1$

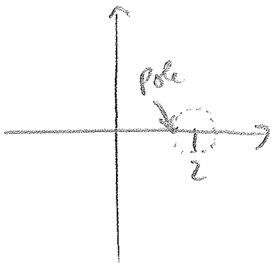
ii) $\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n$ geo series, $|z-1| < 1$

3) $\frac{1}{2z-3}$ at $z_0 = 2$

i) $\left(\frac{d}{dz}\right)^n \left(\frac{1}{2z-3}\right) \Big|_{z=2} = \frac{(-1)^n n!}{2\left(z-\frac{3}{2}\right)^{n+1}} \Big|_{z=2} = \frac{(-1)^n n!}{2\left(\frac{1}{2}\right)^{n+1}} = (-2)^n n!$

$\rightsquigarrow f(z) = \sum_{n=0}^{\infty} \frac{(-2)^n \cancel{n!}}{\cancel{n!}} (z-2)^n = \sum_{n=0}^{\infty} 2^n (2-z)^n \qquad |z-2| < \frac{1}{2}$

ii) $\frac{1}{2z-3} = \frac{1}{2(z-2)+1} = \frac{1}{1-2(2-z)} = \sum_{n=0}^{\infty} 2^n (2-z)^n$ geo series $|z-2| < \frac{1}{2}$.



! pole at $2z-3=0$, $z=\frac{3}{2}$

Laurent Series

Taylor series exist for functions which are holomorphic on discs. Holomorphic functions on certain other types of regions also have series representations

Examples: $\frac{1}{1-z}$: We have a power series in z for $|z|<1$

What can we do for $|z|>1$? Then $\left|\frac{1}{z}\right| < 1$ and we can use a series in $\frac{1}{z}$:

$$\frac{1}{1-z} = \frac{1}{z(\frac{1}{z}-1)} = -\frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \ \left|\frac{1}{z}\right|<1$$

$$= \sum_{n=1}^{\infty}(-z^{-n})$$

absolutely convergent for $|z|>1$ & divergent for $|z|<1$.

Definition

Let A be an annulus, $A = \{z : R_1 < |z-z_0| < R_2\}$ centred at z_0 (with $0 \le R_1 < R_2 \le \infty$).

A series $\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ converging to $f(z)$ at every $z \in A$ is called a Laurent series for f in A.

(Laurent, 1843)

Note: Sometimes we write $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ (i.e. $a_{-n} = b_n$ for $n \ge 1$)

but we have to remember that this is really the sum of two series.

<u>Theorem 3.7</u> (Laurent) Let f be holomorphic on an annulus A, centred at z_0. Then $f(z)$ has a unique Laurent series expansion $\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ converging absolutely to $f(z)$ at every point of A.

<u>Proof:</u> later, using complex integration (There are formulae for a_n and b_n involving integrals). $\square$

<u>Examples</u> 1) $\frac{1}{1-z}$ has Laurent series $\sum_{n=1}^{\infty}(-z^{-n})$

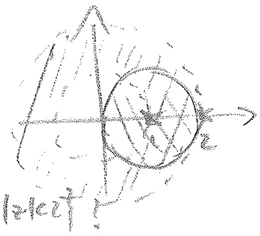
on $A = \{z : 1 < |z| < \infty\}$

and $\sum_{n=0}^{\infty} z^n$ on $A = \{z : 0 < |z| < 1\}$

and, trivially, $(1-z)^{-1}$ on $A = \{z : 0 < |1-z| < \infty\}$

2) $e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$ (by def.) on $A = \{z : 0 < |z| < \infty\}$

3) $f(z) = \frac{1}{(z-1)(z-2)}$



Laurent series on $A = \{z : 1 < |z| < 2\}$?

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})} = \sum_{n=0}^{\infty}\left(-\frac{z^n}{2^{n+1}}\right) + \sum_{n=1}^{\infty}(-z^{-n})$$

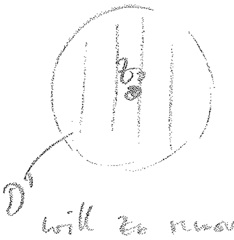
Laurent series on $A = \{z : 0 < |z-1| < 1\}$?

$$f(z) = \frac{1}{z-1}\,\frac{1}{1-(z-1)} = \sum_{n=0}^{\infty}(z-1)^{n-1} = -(z-1)^{-1} + \sum_{n=0}^{\infty}\left[-(z-1)^n\right]$$

Last example: Annulus degenerated to <u>punctured disk</u> $D' = \{z : 0 < |z-z_0| < r\}$

Singularities, Poles, Residues

Definition: If f is holomorphic on a punctured disc $D' = \{z : 0 < |z - z_0| < r\}$ but is not holomorphic, or maybe not defined, at z_0, then z_0 is called an isolated singularity of f.



By Laurent's Theorem, such a f has a Laurent series

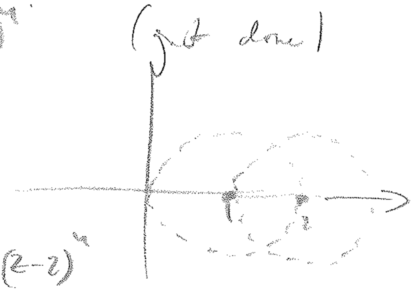
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \quad \text{valid on } D'.$$

Example: $f(z) = \dfrac{1}{(z-1)(z-2)}$ has isolated singularities at $z_0 = 1, 2$

Laurent Series about $z_0 = 1$: $\quad -\dfrac{1}{z-1} + \sum_{n=0}^{\infty} (-1)(z-1)^n \quad$ (just done)

Laurent Series about $z_0 = 2$:

$$\frac{1}{z-2}\,\frac{1}{1-(2-z)} = \frac{1}{z-2}\sum_{n=0}^{\infty}(2-z)^n = \frac{1}{z-2} + \sum_{n=0}^{\infty}(-1)^n (z-2)^n$$



Terminology: If $f(z)$ has an isolated singularity at $z = z_0$ and Laurent Series $\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ then

(i) the part $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ is called the principal part of the Laurent Series (the rest tends to a_0 as $z \to z_0$)

(ii) the coefficient b_1 (of $(z - z_0)^{-1}$) is called the residue of the singularity at z_0 (it has a special role in integration theory).

There are 3 types of isolated singularity:

(a) If the principal part has a finite number of non-zero terms (but doesn't vanish completely), i.e. if it has the form $\sum_{n=1}^{m} b_n (z-z_0)^{-n}$ with $b_m \neq 0$ for some $m \geq 1$ then z_0 is called a <u>pole of order m</u>

A pole of order 1 is called a simple pole

(b) If $b_n \neq 0$ for infinitely many n then z_0 is called an <u>essential singularity</u>

(c) If $b_n = 0 \ \forall n$ (i.e. the principal part is zero) we say that z_0 is a <u>removable singularity</u>. We can then define $f(z_0)$ to be a_0 and $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ will be a Taylor series, valid on the whole disk $D = \{z : 0 \leq |z-z_0| < r\} = \{z : |z-z_0| < r\}$

The type of an isolated singularity can be recognised by the behaviour of f near to it:

(a) If z_0 is a pole then $\lim_{z \to z_0} f(z) = \infty$.

[Obvious: $f(z) = b_m (z-z_0)^{-m} + \ldots \Rightarrow \lim_{z \to z_0} (z-z_0)^m f(z) = b_m$ but $(z-z_0)^m \to 0$, thus $f(z) \to \infty$ as $z \to z_0$]

(b) If z_0 is an essential singularity, then $\lim_{z \to z_0} f(z)$ does not exist

[Not obvious! Even more is true. Picard (1856–1941) proved that in any neighborhood of a isolated essential singularity z_0 f takes on every value in $\mathbb{C}$ with at most one exception.]

(c) If z_0 is removable, $\lim_{z \to z_0} f(z)$ is finite $(= a_0)$ [Obvious]

Examples

1) $e^{1/z}$: isolated singularity at $z=0$. Principal part has infinitely many terms so $z=0$ is an essential singularity

$\left(= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n\right)$

Coeff of z^{-1} is 1 so $\operatorname{Res}_{z=0} e^{1/z} = 1$

[Note: in every neighbourhood of $z=0$, however small, $e^{1/z}$ takes on every value in $\mathbb{C}$ except 0: visible from polar form

$$e^{1/z} = e^{\frac{x}{x^2+y^2}} e^{-i\frac{y}{x^2+y^2}} = re^{i\theta} \Big]$$

2) $f(z) = \frac{1}{(z-1)(z-2)}$ has a isolated singularity at $z=1$

$= -\frac{1}{z-1} + \dots$ simple pole, residue -1

has an isolated singularity at $z=2$

$= \frac{1}{z-2} + \dots$ simple pole, residue $+1$

3) $f(z) = \frac{e^z - 1}{z}$ has a isolated singularity at $z=0$ (undefined)

Laurent series $\frac{1}{z}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots - 1\right) = 1 + \frac{z}{2} + \frac{z^2}{6} + \dots$

Hence the singularity at $z=0$ is removable, define $f(0) = 1$.

Proposition 3.8 If $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where ϕ is holomorphic at z_0 with $\phi(z_0) \neq 0$ and where $m \in \mathbb{N}$, then f has a pole of order m at z_0, residue $\frac{\phi^{(m-1)}(z_0)}{(m-1)!}$

<u>Proof</u> $\phi(z)$ has Taylor series $\sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$ in some neighbourhood of z_0. Hence

$$f(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} = \sum_{n=-m}^{\infty} \frac{\phi^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n$$

a Laurent series with m terms in the principal part. The residue is $(n=-1)$ $\frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ □

<u>Examples</u> 1) $f(z) = \frac{1}{(z-1)(z-2)} = \frac{\phi(z)}{z-2}$ where $\phi(z) = \frac{1}{z-1}$

$\phi(2) = 1 \neq 0 \rightsquigarrow$ simple pole at $z_0 = 2$, residue $\frac{\phi(2)}{0!} = 1$

2) $f(z) = \frac{z+1}{z^2+9}$ singularities at $z = \pm 3i$

$z_0 = 3i$: $f(z) = \frac{\phi(z)}{z-3i}$ where $\phi(z) = \frac{z+1}{z+3i}$

$\phi(3i) = \frac{3i+1}{6i} = \frac{3-i}{6} \neq 0 \rightsquigarrow$ simple pole at $z_0 = 3i$, residue $\frac{3-i}{6}$

(Similarly, simple pole at $z_0 = -3i$, residue $\frac{3+i}{6}$)

3) $f(z) = \frac{z^3+2z}{(z-i)^3}$ singularity at $z_0 = i$

$f(z) = \frac{\phi(z)}{(z-i)^3}$ $\phi(z) = z^3+2z$ $\phi(i) = i \neq 0$

pole of order 3 at $z=i$, residue $\frac{\phi''(i)}{2!} = \frac{6i}{2} = 3i$

4) $f(z) = \frac{\sin z}{z^4}$ has a pole of order 3 (not 4) at $z_0 = 0$

$$f(z) = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} + \dots$$

Residue $\mathrm{res}(f, z=0) = -\frac{1}{6}$.

<u>Note</u>: The importance of residues will become clearer when doing integration: observe that integrating z^{-1} is the "tricky" bit.

Zeros & Poles

Definition If $f(z) = (z-z_0)^m g(z)$ where $m \in \mathbb{N}$ & $g(z_0) \neq 0$

we say f has a zero of order m at z_0

Example: $z(e^z - 1) = z^2 \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)$ has a zero of order 2 at $z_0 = 0$

Observe that f has a zero of order m at z_0 iff

$f(z_0), \dots f^{(m-1)}(z_0)$ are all 0 but $f^{(m)}(z_0) \neq 0$

Proposition 3.9

If p, q are holomorphic at z_0 & $p(z_0) \neq 0$ then $f = p/q$

has a pole of order m at z_0 iff q has a zero of order m at z_0

Proof: "$\Rightarrow$" f pole of order m at z_0

$\Rightarrow f(z) = \frac{\phi(z)}{(z-z_0)^m}$, ϕ holomorphic at z_0, $\phi(z_0) \neq 0$

$\Rightarrow q(z) = (z-z_0)^m \frac{p(z)}{\phi(z)}$ with $\frac{p(z_0)}{\phi(z_0)} \neq 0$

$\Rightarrow q$ has a zero of order m at z_0

"$\Leftarrow$" analogous. $\square$