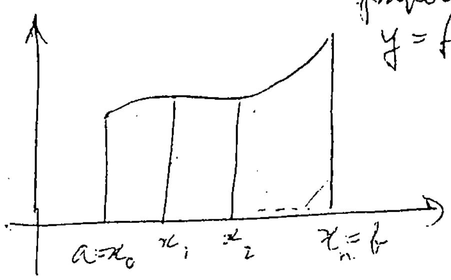
4. Integration

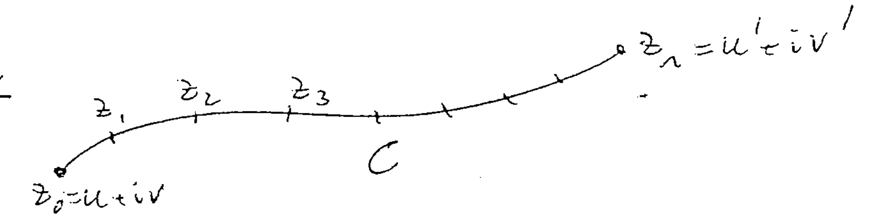
Recall how we define the integral of a real function:



$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \max|x_j - x_{j-1}|\to 0}} \sum_{j=1}^{\infty} f(x_j)(x_j - x_{j-1})$$

In the complex case what will take the place of the partition $x_0 < x_1 \ldots < x_n$?

<u>Basic idea</u>



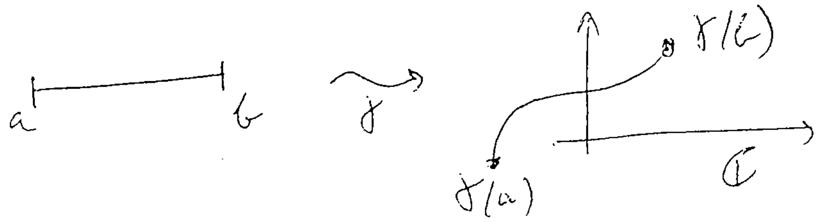
For a given <u>curve</u> C from $u+iv$ to $u'+iv'$ we define

$$\int_C f(z)\,dz = \lim_{\substack{n\to\infty \\ \max|z_j - z_{j-1}|\to 0}} \sum_1^n f(z_j)(z_j - z_{j-1})$$

In practice we shall use an alternative definition which makes calculation easier, but is equivalent to the above. First we need to know more about paths & curves.

<u>Definition</u> A path is a continuous map $\gamma : [a, b] \to \mathbb{C}$, $t \longmapsto \gamma(t) = x(t) + iy(t)$

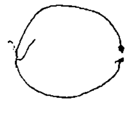
where $[a, b]$ is a closed interval in $\mathbb{R}$



We call the image of γ in $\mathbb{C}$ a <u>curve</u>.

<u>Examples</u> $\gamma_1 : [0,1] \to \mathbb{C}$ $\quad \gamma_1(t) = e^{2\pi i t}$

$\gamma_2 : [0, 2\pi] \to \mathbb{C}$ $\quad \gamma_2(t) = e^{it}$



These define the same curve, but use different parametrisations.
Another useful example is the straight line from w to w':-

$\gamma_3 : [0, 1] \to \mathbb{C}$ $\quad \gamma_3(t) = tw' + (1-t)w$

4. Integration

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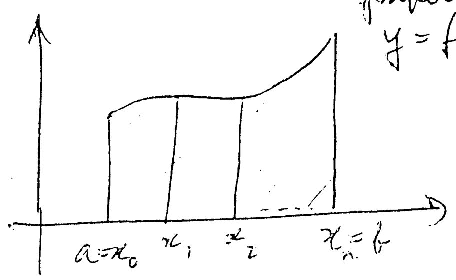

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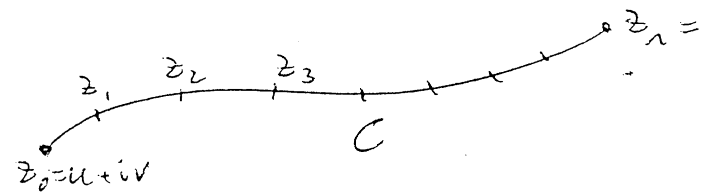

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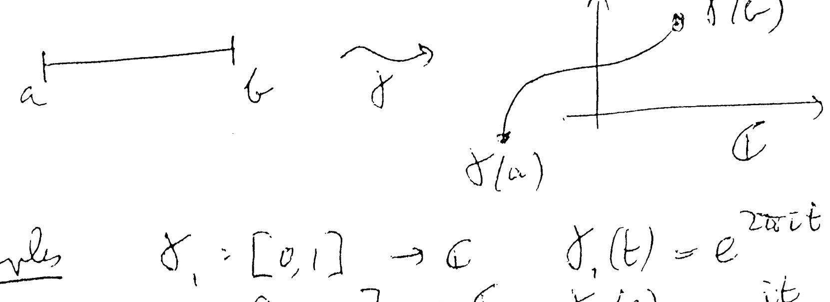

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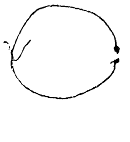

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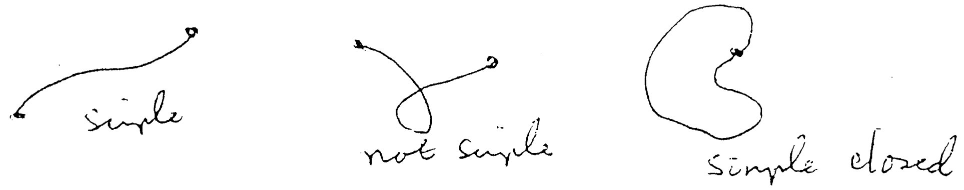
Another useful example is the straight line from w to w':-

$\gamma_3 : [0,1] \to \mathbb{C}$ $\quad \gamma_3(t) = tw' + (1-t)w$

C is a <u>simple</u> curve if (for some parametisation γ of C)

$$t_1 \neq t_2 \implies \gamma(t_1) \neq \gamma(t_2) \quad (\text{i.e. } \gamma \text{ inj})$$

C is a <u>simple closed</u> curve if is parametised by $\gamma : [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$ but $\gamma(t_2) \neq \gamma(t_1)$ for all other $t_2 \neq t_1$

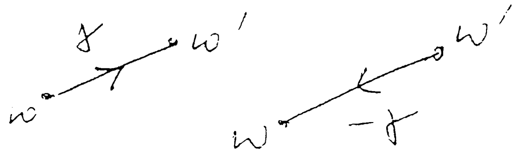


<u>Aside</u>

Jordan Curve Theorem: any simple closed curve divides $\mathbb{C}$ into two regions ("inside" & "outside"): the proof is surprisingly deep & hard

We shall write $-\gamma$ for the <u>negative</u> of γ (i.e. the same path in the opposite direction). Thus if $\gamma : [0,1] \to \mathbb{C}$, $t \mapsto tw' + (1-t)w$

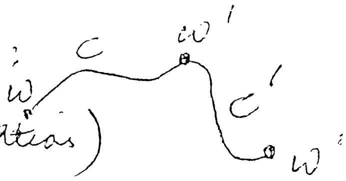
then $(-\gamma) : [0,1] \to \mathbb{C}$, $t \mapsto (1-t)w' + tw$



We can add paths $\gamma_1 : [a,b] \to \mathbb{C}$ & $\gamma_2 : [b,c] \to \mathbb{C}$ in the obvious way:

$$\gamma_1 + \gamma_2 : [a,c] \to \mathbb{C} \qquad t \mapsto \begin{cases} \gamma_1(t) & a \le t \le b \\ \gamma_2(t) & b \le t \le c \end{cases}$$

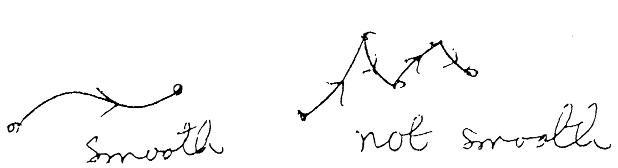
Sometimes we shall add curves:
(by choosing appropriate parametisations)



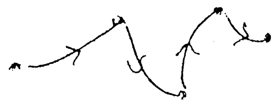
<u>Definition</u>

If C is parametised by $\gamma : [a,b] \to \mathbb{C}$ with γ differentiable on $[a,b]$* & having continuous derivative on $[a,b]$ with $\gamma'(t) \neq 0$ on (a,b) then C is called a <u>smooth</u> curve.

* i.e. $\gamma(t) = x(t) + iy(t)$ has real part $x(t)$ & imaginary part $y(t)$ both differentiable fns of t.



A <u>contour</u> is a piecewise-smooth curve (i.e. a finite union of smooth curves, joined end to end).



<u>N.B.</u> "contour", "curve", "path" are all used in different ways by different authors.

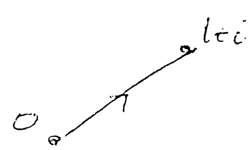
End wk 7.

The <u>length</u> of a smooth curve (or contour) C is

$$L = \int_a^b |\gamma'(t)|\,dt \quad \left(\begin{array}{l}\text{where } \gamma : [a,b] \to C \text{ is a smooth} \\ \text{parametrisation of } C.\end{array}\right)$$

[Think of $\gamma'(t)$ as speed: length = distance travelled]

<u>Example</u> $\gamma : [0,1] \to \mathbb{C}$ $\quad t \mapsto (1+i)t$



$$\text{length} = \int_0^1 \sqrt{1+1}\,dt = \sqrt{2}$$

Length is indept. of parametrisation chosen (not proved here but follows from "length = distance travelled")

<u>Integrating a complex function of a real variable</u>

Given a <u>complex</u> function $w(t) = u(t) + iv(t)$ of a <u>real</u> variable t we <u>define</u>

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

whenever the two integrals on the rhs exist (e.g. if u, v are both continuous, or if they are piecewise continuous). From standard results of real analysis we have:

<u>Proposition 4.1</u> (i) For any $c \in [a,b]$ $\quad \int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$

(ii) $\int_a^b (w_1 + w_2)(t)\,dt = \int_a^b w_1(t)\,dt + \int_a^b w_2(t)\,dt$

(iii) For any $\alpha \in \mathbb{C}$, $\quad \int_a^b \alpha w(t)\,dt = \alpha \int_a^b w(t)\,dt$

(iv) If $W(t)$ is an antiderivative of $w(t)$ (i.e. $W'(t) = w(t)$)
then $\int_a^b w(t)\,dt = W(b) - W(a)$

Note (ii) & (iii) say that $\int_a^b$ is a linear map: $\left\{\begin{matrix}\text{v. space of}\\ \text{complex fns}\\ \text{of a real}\\ \text{variable}\end{matrix}\right\} \to \mathbb{C}$

Example using (iv): to compute $\int_0^{\pi/4} e^{it}\,dt$.

e^{it} has antideriv $\frac{1}{i}e^{it}$ so $\int = \left[\frac{1}{i}e^{it}\right]_0^{\pi/4} = \frac{1}{i}\left(e^{\frac{i\pi}{4}}-1\right) = \frac{1}{\sqrt{2}} + \left(1-\frac{1}{\sqrt{2}}\right)i$.

The following is a very useful estimate for the size of an $\int$ of a complex function of a real variable:

Prop 4.2 $\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$

(Pf omitted, but it comes from the fact that for any finite set of complex nos $\left|\sum_1^n z_j\right| \le \sum_1^n |z_j|$ (Δ-inequality) and the defn. of $\int$ as a limit of finite sums.)

Contour integrals

Let C be a contour & f a continuous complex-valued function defined on C. Suppose C is parametrized by $\gamma : [a,b] \to \mathbb{C}$. Then

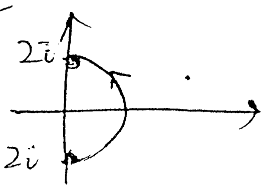
Definition
The contour integral of f along C is $\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

Comment The basic idea is that writing $z = \gamma(t)$ we have $dz = \frac{d\gamma}{dt}.dt$ so that $\int_C f(z)\,dz$ "should be" $\int_a^b f(\gamma(t))\,\frac{d\gamma}{dt}\,.dt$.
This basic idea can be made rigorous: one can prove that the expression $\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$ gives the same value as the definition $\lim_{\substack{n\to\infty\\ |z_j - z_{j-1}|\to 0}} \sum_1^n f(z_j)(z_j - z_{j-1})$ at the start of the chapter. And it is <u>much</u> easier to use in practice.

Examples

1) C semicircle $z=\gamma(\theta)=2e^{i\theta}$ $\quad -\pi/2 \le \theta \le \frac{\pi}{2}$

for. $f(z)=\bar{z}$



$$\int_C f(z)\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\overline{2e^{i\theta}}\right) . 2ie^{i\theta}\,d\theta$$

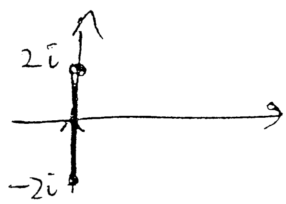
$$= \int_{-\pi/2}^{\pi/2} 4i\,d\theta = 4\pi i$$

2) Same C but for $f(z)=z^2$

$$\int_C z^2\,dz = \int_{-\frac{\pi}{2}}^{\pi/2} (2e^{i\theta})^2\, 2ie^{i\theta}\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8ie^{3i\theta}\,d\theta$$

$$= \left[\tfrac{8}{3} e^{3i\theta}\right]_{-\pi/2}^{\pi/2} = -\frac{16i}{3}$$

3) $f(z)=z^2$ but this time C = straight line path from $-2i$ to $2i$



$$\gamma(t) = 2it + (1-t)(-2i)$$
$$= -2i + 4it$$

$$\int_C z^2\,dz = \int_0^1 (-2i+4it)^2 . 4i\,dt = \int_0^1 4i\left(-4+16t-16t^2\right)dt$$

$$= 4i\left(-4t+8t^2-\frac{16t^3}{3}\right)_0^1 = 4i . \frac{-4}{3} = -\frac{16i}{3}$$

[same as 2)!]

Applying (4.1) to the defn. of $\int_C f(z)\,dz$ we get:

Prop 4.3 (i) $\displaystyle\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$ & $\displaystyle\int_{-C} f\,dz = -\int_C f\,dz$

(ii) $\displaystyle\int_C (f+g)\,dz = \int_C f\,dz + \int_C g\,dz$

(iii) $\displaystyle\int_C \alpha f\,dz = \alpha\int_C f\,dz \quad (\alpha\in\mathbb{C})$

Explain why examples 2 & 3 above same answer!

(iv) If f is continuous on a domain D & has antiderivative F there (i.e. $F'(z)=f(z)\ \forall z\in D$) then $\displaystyle\int_C f\,dz = F(z_1)-F(z_0)$

ends of C

Pf (i)–(iii) obvious from (4.1). For (iv) use $\frac{d}{dt}\left(F(\gamma(t))\right) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\,\gamma'(t)$ $\square$

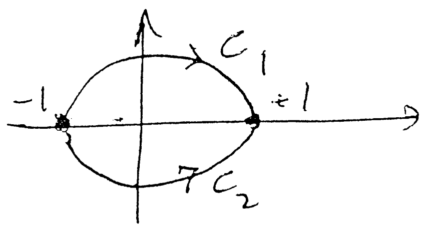
Corollary 4.4 If f is continuous on a domain D and has antiderivative F there, then

(i) If C_1 & C_2 are any two curves both starting at z_0 & ending at z_1, $\int_{C_1} f = \int_{C_2} f$

(ii) If C is a closed contour in D then $\int_C f = 0$.
(sometimes written $\oint_C f$)

Pf
Immediate from 4.3. □

Examples



$$\int_{C_1} \frac{1}{(z-4)^2}\,dz = \int_{C_2} \frac{1}{(z-4)^2}\,dz$$

$$= \left[\frac{-1}{z-4}\right]_{-1}^{+1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

$$\int_{C_1 - C_2} \frac{1}{(z-4)^2}\,dz = 0$$

Our estimate (Prop 4.2) of the size of an integral of a complex integral of a real fn also has an important consequence for contour integrals :-

Prop 4.5 If $|f(z)| \leq M$ $\forall z \in C$ and length$(C) = L$ then
$$\left|\int_C f(z)\,dz\right| \leq ML$$

Proof
$$\left|\int_C f(z)\,dz\right| \underset{\text{defn}}{=} \left|\int_a^b f(\gamma(t)).\gamma'(t)\,dt\right| \underset{4.2}{\leq} \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$
$$\leq M \int_a^b |\gamma'(t)|\,dt \underset{\text{defn}}{=} ML$$

Cauchy's Theorem

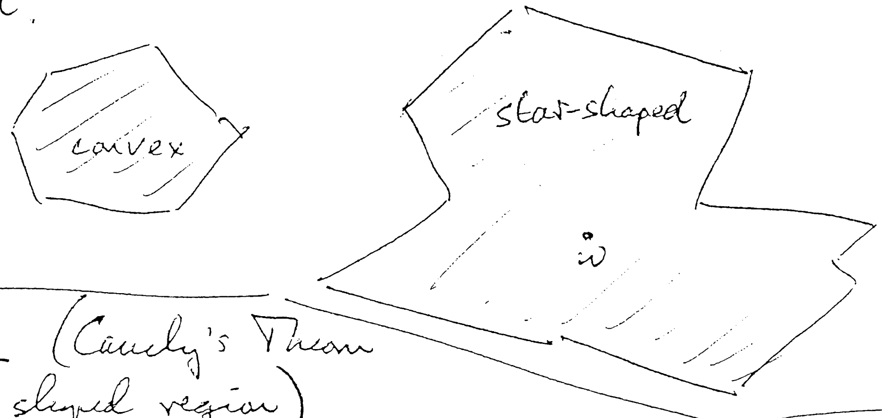
This is the key theorem of complex analysis, so we shall prove it in detail. Afterwards many consequences will follow thick and fast.

Definition

We say $U \subset \mathbb{C}$ is <u>convex</u> if $\forall z_1, z_2 \in U$ the line segment $\{t z_2 + (1-t) z_1 : 0 \leq t \leq 1\} \subset U$

<u>notation</u>: $[z_1, z_2]$

We say U is <u>star-shaped about w</u> if $\forall z \in U$ the line segment $[w, z] \subseteq U$.



<u>Theorem 4.6</u> (Cauchy's Theorem for a star-shaped region)
Let f be a function holomorphic on an open star-shaped region $U \subset \mathbb{C}$. Then for every closed contour C in U, $\int_C f(z)\,dz = 0$

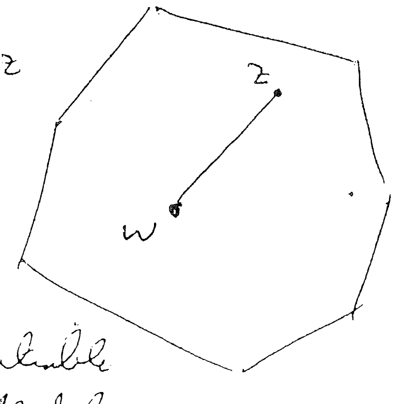
[Cauchy 1789-1857 announced this in 1813 & published a proof in 1825. Gauss knew of the result in 1811. Cauchy's original proof was via Green's Theorem; the proof we look at is essentially due to Goursat and avoids having to assume f' is continuous - which is a <u>consequence</u> of the theorem.]

<u>Proof</u> It will suffice to show:

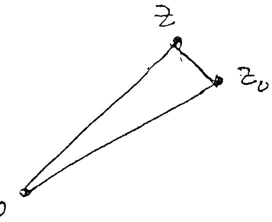
(†) <u>f has an antiderivative F on U</u>, since it will then follow by (4.4) that for a closed curve $\int_C f\,dz = 0$

[We <u>can't</u> use Taylor's Thm since we need Cauchy's Thm before we can <u>prove</u> Taylor's Thm.]

We start by defining $F(z) = \int_{[w,z]} f(z)\,dz$



where w is the "centre" of the region and $[w,z]$ is the straight line segment from w to z. It now just remains to prove that F is differentiable and that $F'(z) = f(z)$. But this will take some work.



$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{\int_{[w,z]} f - \int_{[w,z_0]} f}{z-z_0}$$

Suppose we could show $\int_{[w,z]} f - \int_{[w,z_0]} f = \int_{[z_0,z]} f$ .... (*)

Then $\dfrac{F(z)-F(z_0)}{z-z_0} = \dfrac{\int_{[z_0,z]} f}{z-z_0}$

Now, by continuity of f, given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(\zeta) - f(z_0)| < \varepsilon$ whenever $|\zeta - z_0| < \delta$. Hence, if $|z-z_0| < \delta$ we have

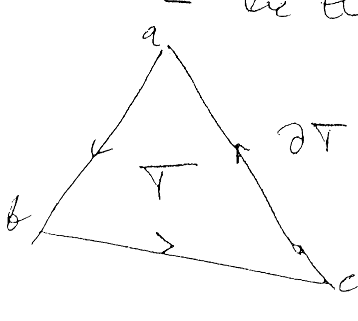
$$\left| \int_{[z_0,z]} f(\zeta)\,d\zeta - \int_{[z_0,z]} f(z_0)\,d\zeta \right| < \varepsilon |z-z_0| \quad (\text{by } 4.5)$$

i.e. $$\left| \int_{[z_0,z]} f(\zeta)\,d\zeta - f(z_0)(z-z_0) \right| < \varepsilon |z-z_0| \quad (\text{if } |z-z_0|<\delta)$$

$$\therefore \quad \left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right| < \varepsilon \quad (\text{if } |z-z_0|<\delta)$$

$\therefore$ F is differentiable at z_0, with derivative $F'(z_0) = f(z_0)$, proving (†) & hence Cauchy's Thm.

It remains only to prove (*), Cauchy's Theorem for a Triangle. Let T be a triangle in U, vertices $\{a,b,c\}$, and let L be the length of its perimeter ∂T. Write $\eta(T) = \int_{\partial T} f$



$$\boxed{\text{We now show that } \int_{\partial T} f = 0}$$

Proof not for exam

"(*)"

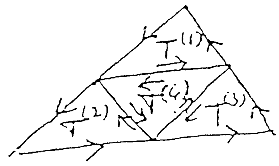
"Cauchy's Theorem for a triangle"

Let f be holomorphic on a domain $\mathcal{U}$ containing a triangle T and its interior. Let ∂T denote the perimeter of the triangle T.

Then $\displaystyle\int_{\partial T} f(z)\,dz = 0$.

Proof Let $\int_{\partial T} f(z)\,dz = \eta(T)$ and let the length of ∂T be L.

Divide T into four triangles by bisecting the sides of T:-



$$\eta(T) = \sum_{j=1}^{4} \eta(T^{(j)}) \quad \text{(internal edges cancel)}$$

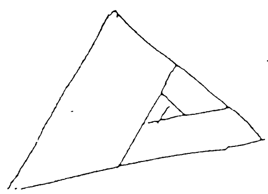
Also each $T^{(j)}$ has $\text{length}(\partial T^{(j)}) = L/2$.

At least one of the $T^{(j)}$ must have $|\eta(T^{(j)})| \geq \frac{1}{4}|\eta(T)|$. Call this triangle T_1.

Now divide T_1 into four triangles in the same way, and deduce that one of these triangles, say T_2, has $|\eta(T_2)| \geq \frac{1}{4}|\eta(T_1)|$.

Note that $\text{length}(\partial T_2) = L/4$.

Repeat, to find $T_1 \supset T_2 \supset T_3 \supset \dots \supset T_n \supset \dots$ with $|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T)|$ and $\text{length}(\partial T_n) = L/2^n$.



$T \supset T_1 \supset T_2 \supset \dots$

Let z_0 be the point $\bigcap_{n=1}^{\infty} T_n$.

Since f is differentiable at z_0, given any $\varepsilon > 0$ we know that $\exists \delta > 0$ such that for $|z - z_0| < \delta$ we have

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq |z - z_0|\,\varepsilon$$

Take n such that $L/2^n < \delta$. Then $|z - z_0|\varepsilon \leq \frac{L\varepsilon}{2^n}$ $\forall z \in \partial T_n$

so $$\left|\int_{\partial T_n} \big(f(z) - f(z_0) - (z - z_0)f'(z_0)\big)\,dz\right| \leq \frac{L\varepsilon}{2^n} \cdot \frac{L}{2^n} = \frac{\varepsilon L^2}{4^n}$$

But $\int_{\partial T_n} \big(f(z_0) + (z - z_0)f'(z_0)\big)\,dz = 0$ since $f(z_0) + (z - z_0)f'(z_0)$ has antiderivative $z f(z_0) + \left(\frac{z^2}{2} - z z_0\right)f'(z_0)$ and ∂T_n is a closed curve.

Hence $\left|\int_{\partial T_n} f(z)\,dz\right| \leq \frac{\varepsilon L^2}{4^n}$. That is, $|\eta(T_n)| \leq \frac{\varepsilon L^2}{4^n}$.

But $|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T)|$. Hence $|\eta(T)| \leq \varepsilon L^2$. But $\varepsilon > 0$ is arbitrary.

Hence $\eta(T) = 0$. $\square$

Note: From the statement of Cauchy's Thm for a star-shaped region one can deduce it for more general regions made up of star-shaped pieces:

$$\int_{C_1+C_2} f = \int_{C_1} f + \int_{C_2} f = 0$$

In particular one can prove that if C is any simple closed contour in $\mathbb{C}$ & f is holo. <u>everywhere on and inside C</u>, then $\int_C f(z)\,dz = 0$ (the <u>general form</u> of Cauchy's Thm)

<u>Corollary 4.7</u> (Deformation Principle)

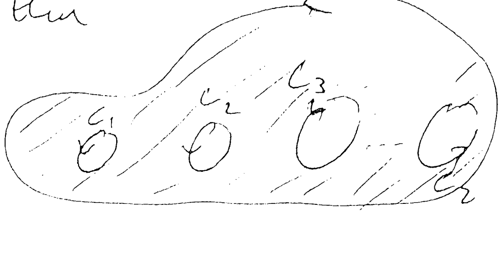
If f is holo. on the region between 2 simple closed contours, disjoint from each other, then
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

<u>Proof</u> (idea) Divide up the region between C_1 & C_2, by "cuts," into star-shaped regions:

Now $\int_{C_1} f\,dz - \int_{C_2} f\,dz =$ sum of integrals around closed contours shown $= 0$ by (4.6) □

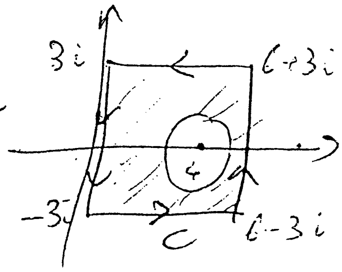
More generally, the same proof gives:

<u>Corollary 4.8</u> If C is a simple closed contour & $C_1, \dots, C_n$ are simple closed contours inside C, with disjoint interiors, and f is holo on the region between C & the C_j, then
$$\int_C f\,dz = \int_{C_1} f\,dz + \dots + \int_{C_n} f\,dz$$



Example (of deformation principle)

To find $\int_C \frac{z}{(z-4)^2}\,dz$ where C is as shown



$\int_C = \int_{C'}$ where C' has centre 4, radius 1

So C' is parametrised by $\gamma : t \longmapsto 4 + e^{2\pi i t}$ $(0 < t < 1)$

Hence $\int_{C'} \frac{z}{(z-4)^2}\,dz = \int_0^1 \frac{4+e^{2\pi i t}}{e^{4\pi i t}} \cdot 2\pi i\, e^{2\pi i t}\,dt$

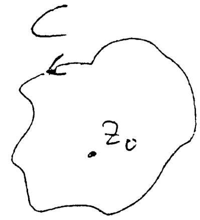
$$= 2\pi i \int_0^1 \left(4e^{-2\pi i t} + 1\right) dt = 2\pi i \left[-\frac{2}{\pi i} e^{-2\pi i t} + t\right]_0^1 = 2\pi i$$

Proposition 4.9 (The Cauchy Integral Formula)

If ϕ is holomorphic on and everywhere inside a simple closed contour C, and if z_0 is inside C then (if C is parametrised in the positive, i.e. anticlockwise, direction)

$$\phi(z_0) = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{z - z_0}\,dz$$

Note This tells us that once we know the value of ϕ at every point *on* C, we can compute its value at every point *inside* C.



We shall use the C.I.F. in both directions – to compute $\phi(z_0)$ using $\int_C$ and vice versa.

Proof of 4.9: ϕ is continuous so given any $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |\phi(z) - \phi(z_0)| < \varepsilon$. Choose R s.t. $R < \delta$ and the disc centre z_0 radius R is inside C. Let C_0 be the boundary of this disc.

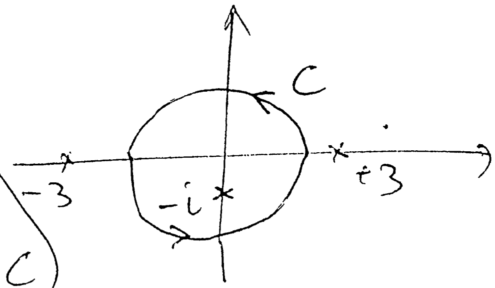
$$\int_C \frac{\phi(z)}{z-z_0}\,dz \underset{4.7}{=} \int_{C_0} \frac{\phi(z)}{z-z_0}\,dz = \int_{C_0} \frac{\phi(z_0)}{z-z_0}\,dz + \int_{C_0} \frac{\phi(z)-\phi(z_0)}{z-z_0}\,dz$$

$$= \phi(z_0) \int_0^{2\pi} \frac{1}{Re^{i\theta}} \cdot Rie^{i\theta}\,d\theta + I = 2\pi i\,\phi(z_0) + I$$

where $|I| \le \max_{z \in C_0} \left|\frac{\phi(z)-\phi(z_0)}{z-z_0}\right| \times \text{length } C_0 < \frac{\varepsilon}{R} \cdot 2\pi R$. But ε is arb. So $I = 0$ □

Example Let C be the pos. or. circle $|z|=2$. Calculate

$$\int_C \frac{z}{(9-z^2)(z+i)}\,dz \ ?$$



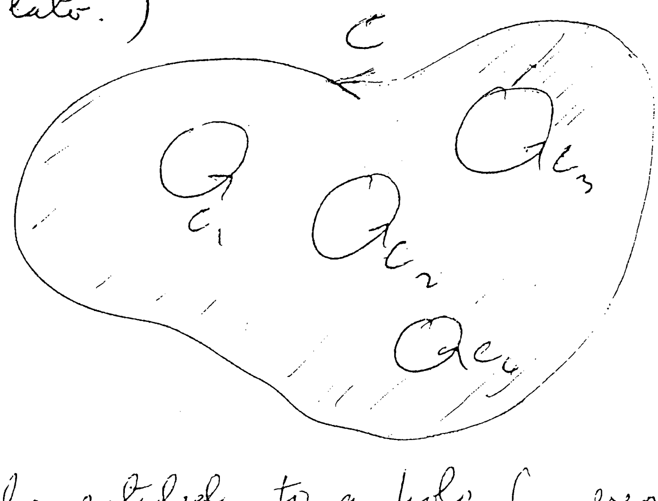
$$\int = \int_C \frac{\phi(z)}{z+i}\,dz \quad \left(\begin{array}{l}\text{where } \phi(z)=\frac{z}{9-z^2} \\ \text{holo. on \& inside } C\end{array}\right)$$

$$\underset{\text{C.I.F.}}{=} 2\pi i\,\phi(-i) = 2\pi i \cdot \frac{-i}{9-(-i)^2} = \frac{\pi}{5}$$

Theorem 4.10 (The Residue Theorem) Let f be holo. on & inside the simple closed contour C, except at a finite no. of singularities $z_1, \dots, z_n$, all inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f$$

Proof (in the case $z_1, \dots, z_n$ simple poles or removable singularities: we shall prove the general case later.)



Draw small circles around the singularities $z_1, \dots, z_n$.

By Corollary 4.8

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

If z_k is removable then f can be extended to a holo. fn everywhere inside C_k, so $\int_{C_k} f(z)\,dz = 0$ (Cauchy's Thm): but $\operatorname{Res}_{z_k} f = 0$ too.

If z_k is a simple pole, then $f(z) = \frac{\phi(z)}{z-z_k}$ with ϕ holo. at z_k; and hence $\int_{C_k} f(z)\,dz = \int_{C_k} \frac{\phi(z)}{z-z_k}\,dz = 2\pi i\,\phi(z_k)$ (C.I.F.). But $\operatorname{Res}_{z_k} f = \phi(z_k)$ too. $\square$

Remark For the general case we shall need the technical results that f has a Laurent series and that $\int\left(\sum_0^{\infty} a_n (z-z_0)^n + \sum_1^{\infty} b_n (z-z_0)^{-n}\right) dz$
$= \sum_0^{\infty} a_n \int_C (z-z_0)^n\,dz + \sum_1^{\infty} b_n \int_C (z-z_0)^{-n}\,dz$. Once we have this we are OK

4.13

Since $\int_C (z-z_0)^n\,dz = 0 \quad \forall n \neq -1$ (by Cauchy's Thm)

& $\int_C (z-z_0)^{-1}\,dz = 2\pi i$ (by direct computation or Cauchy's Integral formula)

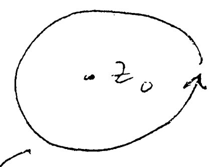
See Chapt 5 for the technical detail.

Why does $\frac{1}{z-z_0}$ have no antiderivative? The problem is that $\log(z-z_0)$ is multi-valued:

$$\log(z-z_0) = w \iff e^{w} = z-z_0$$
$$\iff e^{u+iv} = z-z_0$$
$$\iff e^{u} = |z-z_0| \ \& \ v = \arg(z-z_0)$$
$$\iff w = \ln|z-z_0| + i\arg(z-z_0)$$

Thus $\log(z-z_0)$ is only defined up to $\pm 2\pi n i$

We can make log a single-valued function by choosing its principal value, but then we get a discontinuity when we travel once around a circle centred at z_0.

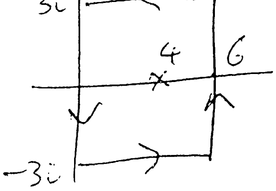


$\cdot z_0$ $\arg(z-z_0)$ increases by 2π in one circuit of C.

C

Examples applying the Residue Theorem

1) $\int_C \frac{z}{(z-4)^2}\,dz$ (C as shown)



(as before!)

$\frac{z}{(z-4)^2}$ has a pole, order 2, at $z=4$ but is holo elsewhere.

Here $\int_C \frac{z}{(z-4)^2}\,dz = 2\pi i \operatorname{Res}_{z=4} \frac{z}{(z-4)^2} = 2\pi i \frac{\phi'(4)}{1!}$ (where $\phi(z)=z$)

$= 2\pi i$

2) $\int_C \frac{1}{(1-z)(2-z)}\,dz$ Poles order 1 at $z=1$ (Res -1)

& order 1 at $z=2$ (Res $+1$)

So for simple closed curve C, $\int_C \frac{1}{(1-z)(2-z)}\,dz = \begin{cases} -2\pi i & \text{if } C \text{ contains } z=1 \text{ but not } z=2 \\ +2\pi i & \text{if } C \text{ contains } z=2 \text{ (but not } z=1) \\ 0 & \text{otherwise} \end{cases}$

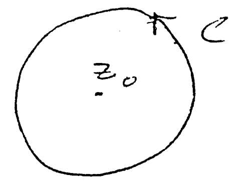
Theorem 6.11 If ϕ is holo. at z_0 then the derivatives of ϕ of all orders exist and are holomorphic at z_0.

Note There is nothing like this for real functions! Differentiable once $\not\Rightarrow$ differentiable twice!

Sketch proof of 6.11

Choose a small circle C around z_0 s.t. ϕ is holo on & inside C
(Recall ϕ holo at z_0 means differentiable at every pt in some neighbourhood of z_0).

By the Cauchy integral formula $\phi(z_0) = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{z-z_0}\,dz$



"Differentiating under the $\int$, with respect to z_0" gives:

$$\phi'(z_0) = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{(z-z_0)^2}\,dz$$

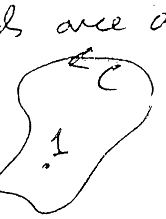
$$\phi''(z_0) = \frac{2!}{2\pi i} \int_C \frac{\phi(z)}{(z-z_0)^3}\,dz$$

$$\phi^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{\phi(z)}{(z-z_0)^{n+1}}\,dz. \qquad \square$$

Corollary 6.12 (Cauchy's Integral Formula - Extended Version)

Let ϕ be holo on & inside the simple closed contour C, & z_0 be inside C. Then $\forall n \geq 0$
$$\phi^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{\phi(z)}{(z-z_0)^{n+1}}\,dz \qquad \square$$

Example Compute $\int_C \frac{z^3}{(z-1)^4}\,dz$ where C winds once around $z=1$



Set $\phi(z) = z^3$.

Apply 6.12 with $n=3$

$$\phi^{(3)}(1) = \frac{3!}{2\pi i} \int_C \frac{\phi(z)}{(z-1)^4}\,dz \qquad \therefore \int = \frac{2\pi i}{3!}\phi^{(3)}(1) = \frac{2\pi i}{3!} \times 6 = 2\pi i$$

[Or use residue thm. But to compute $\operatorname{Res}_1 \frac{z^3}{(z-1)^4}$ involves exactly the same calculation.]