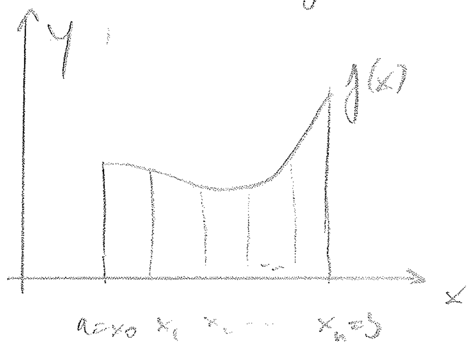
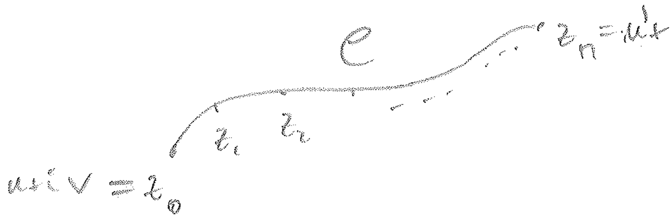
4. Integration

Recall how we define the integral of a real function



$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \max|x_j - x_{j-1}|\to 0}} \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$

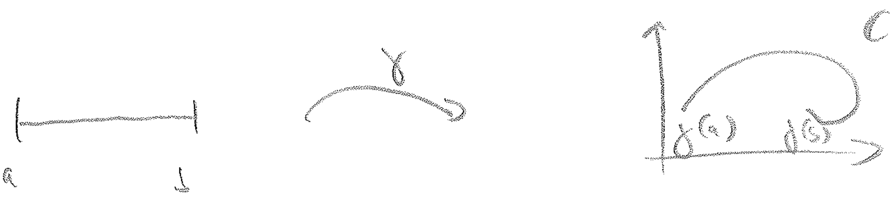
In the complex case, the integral along a real interval from a to b is replaced by the integral along a curve C from $u+iv$ to $u'+iv'$



$$\int_C f(z)\,dz = \lim_{\substack{n\to\infty \\ \max|z_j - z_{j-1}|\to 0}} \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1})$$

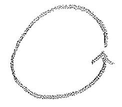
In practice, we shall use an alternative definition which makes calculation easier, and is equivalent to the above. First we need to know more about paths and curves.

Definition A path is a continuous map $\gamma : [a,b] \to \mathbb{C}$, $t \mapsto x(t) + iy(t)$, where $[a,b]$ is a closed interval in $\mathbb{R}$



We call the image of $[a,b]$ under γ a curve.

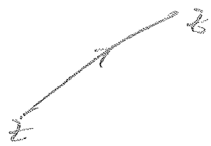
<u>Examples:</u> $\gamma_1 : [0,1] \to \mathbb{C}$ $\quad \gamma_1(t) = e^{2\pi i t}$



$\gamma_2 : [0,2\pi] \to \mathbb{C}$ $\quad \gamma_2(t) = e^{it}$

They define the same curve, but use different parametrisation.
Another useful example is the straight line from z to z':

$\gamma_3 : [0,1] \to \mathbb{C}$ $\quad \gamma_3(t) = t z' + (1-t) z$



$\mathcal{C}$ is a <u>simple</u> curve if (for some parametrisation γ of $\mathcal{C}$)

$$t_1 \neq t_2 \Rightarrow \gamma(t_1) \neq \gamma(t_2) \qquad \text{(i.e. } \gamma \text{ injective)}$$

$\mathcal{C}$ is a <u>simple closed</u> curve if it is parametrisable by $\gamma : [a,b] \to \mathbb{C}$

with $\gamma(a) = \gamma(b)$ but $\gamma(t_1) \neq \gamma(t_2)$ for all other $t_1 \neq t_2$.

[Aside: Jordan Curve Theorem: any simple closed curve divides $\mathbb{C}$ into two regions ("inside" and "outside"). Seemingly obvious, but the proof is surprisingly deep & hard]

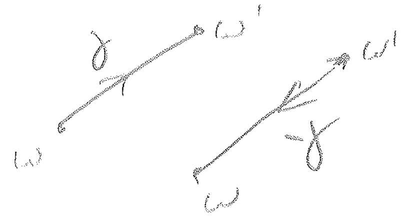
We shall write $-\gamma$ for the <u>negative</u> of γ (i.e. the same path in the opposite direction). Thus, if $\gamma : [0,1] \to \mathbb{C}$, $t \mapsto \gamma(t)$

then $(-\gamma) : [0,1] \to \mathbb{C}$

$$t \mapsto (-\gamma)(t) = \gamma(1-t)$$

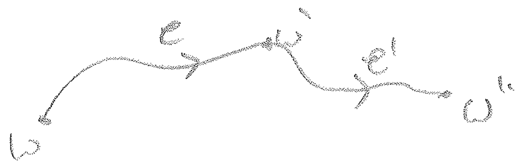
e.g. $\gamma(t) = t\omega' + (1-t)\omega$

$$(-\gamma)(t) = (1-t)\omega' + t\omega$$



We can add paths $\gamma_1 : [a,b] \to \mathbb{C}$ & $\gamma_2 : [b,c] \to \mathbb{C}$ in the obvious way: $\gamma_1 + \gamma_2 : [a,c] \to \mathbb{C}$, $t \mapsto \begin{cases} \gamma_1(t) & a \le t \le b \\ \gamma_2(t) & b \le t \le c \end{cases}$

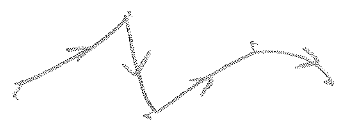
sometimes we shall add curves (by choosing appropriate parametrisation)



<u>Definition</u> If C is parametrized by $\gamma : [a,b] \to \mathbb{C}$ with γ differentiable on $[a,b]^*$ and having continuous derivative on $[a,b]$ with $\gamma'(t) \neq 0$ on $[a,b]$ then C is called a <u>smooth</u> curve.

* i.e. $\gamma(t) = x(t) + iy(t)$ with real part $x(t)$ and imag. part $y(t)$ both diff'able functions of t.

A <u>contour</u> is a piecewise smooth curve (i.e. a finite union of smooth curves, joined end to end)



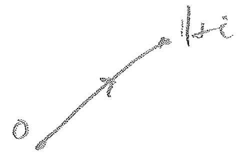
N.B.: "contour", "curve", "path" are all used in different ways by different authors.

The <u>length</u> of a smooth curve (or contour) $\mathcal{C}$ is

$$L = \int_a^b |\gamma'(t)|\, dt \quad \Big(\text{where } \gamma : [a,b] \to \mathbb{C} \text{ is a smooth parametrisation of } \mathcal{C}\Big)$$

[think of $\gamma'(t)$ as speed : length = distance travelled] (19.11.a)

<u>Example</u>: $\gamma : [0,1] \to \mathbb{C} \qquad t \mapsto (1+i)t$



length $\displaystyle L = \int_0^1 \sqrt{1+1}\, dt = \sqrt{2}$

length is independent of parametrisation chosen (not proved here, but follows from "length = distance travelled").

<u>Integrating a complex function of a real variable</u>

Given a <u>complex</u> function $w(t) = u(t) + i\,v(t)$ of a <u>real</u> variable t we <u>define</u>

$$\int_a^b w(t)\, dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

whenever the two integrals on the rhs exist

(e.g. if u, v are both continuous, or if they are piecewise continuous)

From standard results we have

<u>Proposition 4.1</u> (i) For any $c \in [a,b]$

$$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$$

(ii) $$\int_a^b (w_1 + w_2)(t)\,dt = \int_a^b w_1(t)\,dt + \int_a^b w_2(t)\,dt$$

(iii) for any $\alpha \in \mathbb{C}$: $$\int_a^b \alpha w(t)\,dt = \alpha \int_a^b w(t)\,dt$$

(iv) If $W(t)$ is an antiderivative of $w(t)$ (i.e. $W'(t) = w(t)$)

then $$\int_a^b w(t)\,dt = W(b) - W(a)$$

Note: (ii) & (iii) say that $\int_a^b$ is a linear map:

$$\{\text{space of complex functions of a real variable}\} \to \mathbb{C}$$

<u>Example</u> (using (iv)): to compute $\displaystyle\int_0^{\pi/4} e^{it}\,dt$

e^{it} has antiderivative $\frac{1}{i} e^{it}$, so $\displaystyle\int_0^{\pi/4} e^{it}\,dt = \frac{1}{i} e^{it}\Big|_0^{\pi/4} =$

$$= \frac{1}{i}\left(e^{i\pi/4} - 1\right) = \frac{1}{i}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - 1\right) = \frac{\sqrt{2}}{2} + \left(1 - \frac{\sqrt{2}}{2}\right)i$$

The following is a very useful estimate for the size of an integral of a complex function of a real variable:

<u>Proposition 4.2</u> $\quad \left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt$

<u>Proof</u>: omitted, but it comes from the fact that for any finite set of complex #'s

$\left| \sum_{\nu=1}^{n} z_\nu \right| \le \sum_{\nu=1}^{n} |z_\nu|$ ($\triangle$-inequality) and the definition of

$\int$ as a limit of finite sums $\square$

<u>Contour integrals</u>

Let C be a contour & f a continuous complex-valued function defined on C. Suppose C is parametrised by $\gamma : [a,b] \to \mathbb{C}$. Then

<u>Definition</u>

$$\boxed{\text{the } \underline{\text{Contour integral}} \text{ of } f \text{ along } C \text{ is} \quad \int_C f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt}$$

<u>Comment</u>: The basic idea is that writing $z = \gamma(t)$ we have $dz = \frac{d\gamma}{dt}dt$ so that $\int_C f(z)\,dz$ "should be" $\int_a^b f(\gamma(t))\frac{d\gamma}{dt}\,dt$ *

This can be made rigorous: you can prove that the expression * gives the same value as the defn $\lim_{n\to\infty} \sum$

This can be made rigorous: one can prove that the expression * gives the same value as the definition

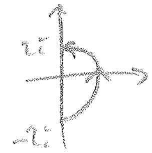
$$\lim_{\substack{n\to\infty \\ |z_j - z_{j-1}|\to 0}} \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1})$$

at the start of the chapter. And it is much easier to use in practice.

Examples

1) C semicircle $\quad z=\gamma(\theta)=2e^{i\theta} \quad -\pi/2<\theta<\pi/2$

pick $f(z)=\bar z \qquad \gamma'(\theta)=2ie^{i\theta}$



$$\int_C f(z)\,dz = \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}}\, 2ie^{i\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} 4i\, d\theta = 4\pi i$$

19.11.5

2) same C but $f(z)=z^2$:

$$\int_C f(z)\,dz = \int_{-\pi/2}^{\pi/2} 4e^{2i\theta}\, 2ie^{i\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} 8i e^{3i\theta}\, d\theta = \frac{8}{3} e^{3i\theta}\Big|_{-\pi/2}^{\pi/2} = -\frac{16}{3} i$$

3) $f(z)=z^2$ but $\tilde C$ straight path from $-2i$ to $2i$: $\gamma(t)=-2i+4it$, $0\le t$

$$\int_{\tilde C} f(z)\,dz = \int_0^1 (-2i+4it)^2\, 4i\, dt = \int_0^1 16i(-1+4t-4t^2)\,dt$$

$$= -16i\left(t-2t^2+\frac{8}{3}t^3\right)\Big|_0^1 = -\frac{16}{3} i \qquad \text{(same as 2) !)}$$

Applying (4.1) to the definition of $\int_C f(z)\,dz$ we get:

Prop 4.3 (i) $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz, \quad \int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

(ii) $\int_C (f+g)(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

(iii) $\int_C \alpha f(z)\,dz = \alpha \int_C f(z)\,dz \quad (\alpha\in\mathbb{C})$

(iv) If f is continuous on a domain D & has antiderivative F (i.e. $F'(z) = f(z)\ \forall z \in D$) then $\int_C f(z)\,dz = F(z_1) - F(z_0)$ ← ends of C

Proof: (i)–(iii) obvious from (4.1). For (iv) use

$$\frac{d}{dt} F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t) \rightsquigarrow \int_C f(z)\,dz = \int_{t_0}^{t_1} \frac{d}{dt} F(\gamma(t))\,dt \qquad \square$$

Corollary 4.4 If f is continuous on a domain D and has antiderivative F then,

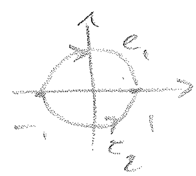
then (i) if C_1 & C_2 are any two curves both starting at z_0 and ending at z_1

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

(ii) if C is a closed contour in D then $\int_C f(z)\,dz = 0$

(sometimes written $\oint_C f(z)\,dz$)

Examples: $$\int_{C_1} \frac{1}{(z-4)^2}\,dz = \int_{C_2} \frac{1}{(z-4)^2}\,dz$$



$$= -\frac{1}{z-4}\Big|_{-1}^{1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \quad ; \quad \int_{C_1 - C_2} \frac{1}{(z-4)^2}\,dz = 0$$

The estimate (4.2) of the size of an integral of a complex fct. of a real variable also has an important consequence for contour integrals.

Proposition 4.5 If $|f(z)| \le M\ \forall z \in C$ and length$(C) = L$ then

$$\left|\int_C f(z)\,dz\right| \le ML$$

Proof $$\left|\int_C f(z)\,dz\right| \underset{\text{def}}{=} \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \underset{(4.2)}{\le} \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\le M \int_a^b |\gamma'(t)|\,dt \underset{\text{def}}{=} ML \qquad \square$$

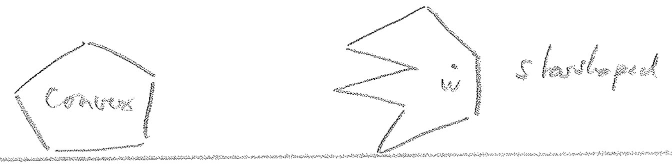
Cauchy's Theorem

This is the key theorem of complex analysis, so we shall prove it in detail. Afterwards many consequences will follow easily.

Definition We say $U \subset \mathbb{C}$ is convex if $\forall z_1, z_2 \in U$ the line segment $\{t z_2 + (1-t) z_1 : 0 \le t \le 1\} \subseteq U$

notation: $[z_1, z_2]$

We say U is star-shaped about w if $\forall z \in U$ the line segment $[w, z] \subseteq U$



> **Theorem 4.6** (Cauchy's Theorem for a star-shaped region)
>
> Let f be a function holomorphic on an open star-shaped region $U \subset \mathbb{C}$. Then for every closed contour C in U $\int_C f(z)\, dz = 0$

Cauchy 1789–1857 announced this in 1813 & published a proof in 1825. Gauß knew of the result in 1811. Cauchy's original proof was via Green's Theorem. The proof look at is essentially due to Goursat, and avoids having to assume continuity of f', which is actually a consequence of the theorem.

Proof It will suffice to show: $\boxed{f \text{ has an antiderivative } F \text{ on } U}$ $(+)$

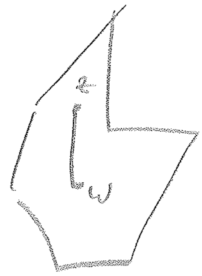
since it will follow by (4.4) that for a closed curve $\int_C f(z)dz = 0$

[We can't use Taylor's Theorem since we will need Cauchy's Theorem to prove Taylor's Theorem]

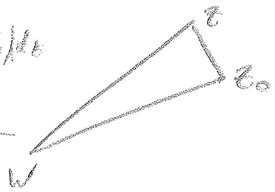
We start by defining $F(z) = \int_{[w,z]} f(z)\,dz$

where w is the ~~dom~~ "centre" of the (star-shaped) region and $[w,z]$ is the straight line segment from w to z. It now just remains to prove that F is differentiable and that $F'(z) = f(z)$.

But this will take some work.



$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{\int_{[w,z]} f(z)\,dz - \int_{[w,z_0]} f(z)\,dz}{z-z_0}$$



Suppose we could show

$$\int_{[w,z]} f(z)\,dz - \int_{[w,z_0]} f(z)\,dz = \int_{[z_0,z]} f(z)\,dz \qquad (*)$$

Then $\dfrac{F(z)-F(z_0)}{z-z_0} = \dfrac{\int_{[z_0,z]} f(z)\,dz}{z-z_0}$ and, by continuity of f,

given $\varepsilon > 0$ $\exists\, \delta > 0$ with $|f(s) - f(z_0)| < \varepsilon$ $\forall$ $|s - z_0| < \delta$

Thus, for $|s - z_0| < \delta$ we have $\left| \int_{[z_0,z]} f(s)\,ds - \underbrace{\int_{[z_0,z]} f(z_0)\,ds}_{f(z_0)(z-z_0)} \right| < \varepsilon\,|z - z_0|$

Therefore,

$$\left| \frac{\int_{[z_0,z]} f(s)\,ds}{z-z_0} - f(z_0) \right| < \varepsilon \quad \text{for } |z - z_0| < \delta$$

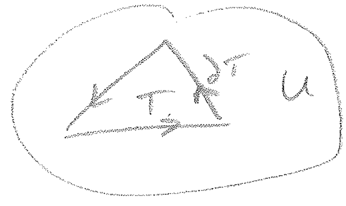
i.e. $\lim_{z \to z_0} \dfrac{F(z)-F(z_0)}{z-z_0} = f(z_0)$, F is differentiable at z_0 with derivative $F'(z_0) = f(z_0)$, proving $(+)$ and Cauchy's theorem.

All we need to do now is prove $(*)$, Cauchy's Theorem for a Δ.

Cauchy's Theorem for a Triangle

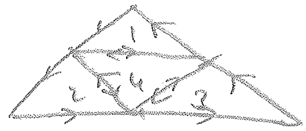
Let f be holomorphic on a domain U containing a triangle T and its interior.

Let ∂T denote the perimeter of the triangle T.



Then $\displaystyle\int_{\partial T} f(z)\, dz = 0$

<u>Proof</u> Let $\eta(T) = \displaystyle\int_{\partial T} f(z)\, dz$ and let the length of ∂T be L. Divide T into four triangles by bisecting the sides of T



$$\eta(T) = \sum_{j=1}^{4} \eta(T^{(j)}) \qquad \text{(internal edges cancel)}$$

Each $T^{(j)}$ has $\text{length}(\partial T^{(j)}) = L/2$. At least one of the $T^{(j)}$ must have $|\eta(T^{(j)})| \geq \frac{1}{4}|\eta(T)|$. Call this triangle T_1. Now repeat this process to get T_2 with $\text{length}(\partial T_2) = L/4$ and $|\eta(T_2)| \geq \frac{1}{4}|\eta(T_1)| \geq \frac{1}{16}|\eta(T)|$

Repeat to get $T \supset T_1 \supset T_2 \supset T_3 \supset \dots$ with $\text{length}(\partial T_n) = L/2^n$ and $|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T)|$. Let z_0 be the point $\displaystyle\bigcap_{n \geq 1}^{\infty} T_n$.

Since f is differentiable at z_0, given any $\varepsilon > 0$ we know that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall |z - z_0| < \delta : |f(z) - f(z_0) - (z - z_0) f'(z_0)| \leq |z - z_0| \varepsilon$$

Take n such that $L/2^n < \delta$. Then $|z - z_0|\varepsilon < \frac{L}{2^n}\varepsilon \quad \forall z \in \partial T_n$

Thus, $\displaystyle\left| \int_{\partial T_n} \big(f(z) - \underbrace{f(z_0) - (z - z_0) f'(z_0)} \big)\, dz \right| \leq \frac{L}{2^n}\varepsilon \frac{L}{2^n} = \varepsilon \frac{L^2}{4^n}$

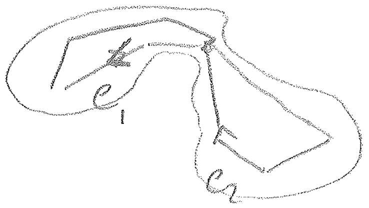
Integral vanishes since antiderivative $-z f(z_0) - \left(\frac{z^2}{2} - z_0 z\right) f'(z_0)$ exists & closed curve ∂T_n

$\displaystyle\leadsto \left| \int_{\partial T_n} f(z)\, dz \right| \leq \varepsilon \frac{L^2}{4^n} \leadsto \frac{\varepsilon L^2}{4^n} \geq |\eta(T_n)| \geq \frac{1}{4^n}\eta(T)$

$\leadsto |\eta(T)| \leq \varepsilon L^2$ but $\varepsilon > 0$ arbitrary. $\Rightarrow \eta(T) = 0$ $\square$

Notes. 1) If we assume f' continuous, then Cauchy's Theorem follows from Stokes' Theorem (Calculus III). But the point here is f differentiable once $\to$ f differentiable many times. Thus, continuity of f' is a <u>consequence</u> of Cauchy's Theorem and there is no need to assume it as a hypothesis.

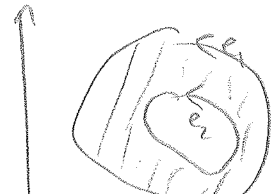
2) From the statement of Cauchy's Theorem for a star-shaped region one can deduce it for more general regions made up of star-shaped pieces



$$\int_C = \int_{C_1} + \int_{C_2}$$

In particular, one can prove that if C is any simple closed contour in $\mathbb{C}$ & f is holomorphic <u>everywhere on and inside C</u> then $\int_C f(z)\,dz = 0$ (the <u>general form</u> of Cauchy's Theorem)

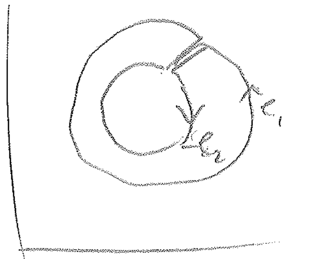
<u>Corollary 4.7 (Deformation Principle)</u>



If f is holomorphic on the region between two simple closed contours, disjoint from each other, then
(and some calculation!)

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

<u>Proof</u> (Idea)



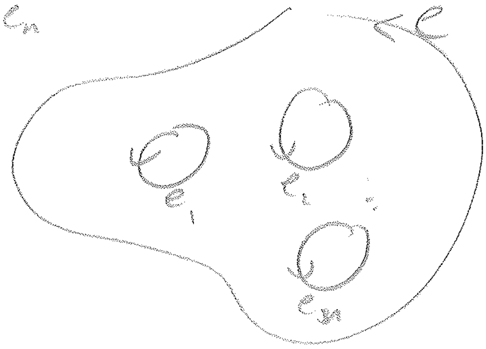
$$0 = \int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

(using the general form of Cauchy's Theorem)

More generally, the same proof gives

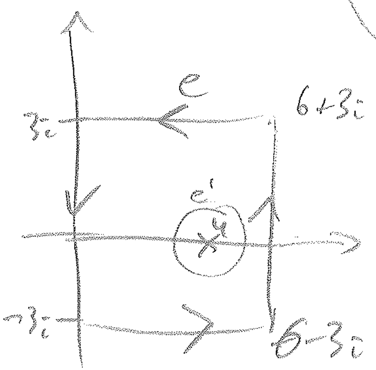
Corollary 4.8 If C is a simple closed contour & $C_1, \dots C_n$ are simple closed contours inside C with disjoint exteriors and same orientation, and f is holomorphic on the region between C and the C_j's, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \dots + \int_{C_n} f(z)\,dz$$



Example (of deformation principle)

To find $I = \int_C \frac{z}{(z-4)^2}\,dz$ where C is



$\int_C = \int_{C'}$ where C' has centre 4 and radius 1. Choose $\gamma'(t) = 4+e^{it}$, $0 \le t \le 2\pi$ to get

$$I = \int_{C'} \frac{z}{(z-4)^2}\,dz = \int_0^{2\pi} \frac{4+e^{it}}{(e^{it})^2}\, ie^{it}\,dt = \int_0^{2\pi} 4ie^{-it} + i\,dt$$

$$= \left(-4e^{-it} + it\right)\Big|_0^{2\pi} = 2\pi i$$

Proposition 4.9 (The Cauchy Integral Formula)

If ϕ is holomorphic on and everywhere inside a simple closed contour C,

and if z_0 is inside C then (if C is parametrised in the positive, i.e. anticlockwise direction)

$$\phi(z_0) = \frac{1}{2\pi i} \oint_C \frac{\phi(z)}{z - z_0}\, dz$$

Note: this tells us that we can compute the value of ϕ at any point inside C once we know its value at every point on C.

We shall use the CIF in both directions – to compute $\phi(z_1)$ using $\int_C$ and vice versa.

Proof of 4.9: ϕ is continuous, so given any $\varepsilon > 0$ $\exists$ $\delta > 0$ with $|z - z_0| < \delta \Rightarrow |\phi(z) - \phi(z_0)| < \varepsilon$. Choose $R < \delta$ such that the disk with radius R about z_0 is inside C. Let C_0 be the boundary of this disk:

$$\int_C \frac{\phi(z)}{z - z_0}\, dz \underset{4.7.}{=} \int_{C_0} \frac{\phi(z)}{z - z_0}\, dz$$

$$= \int_{C_0} \frac{\phi(z_0)}{z - z_0}\, dz + \underbrace{\int \frac{\phi(z_0) - \phi(z)}{z - z_0}\, dz}$$

$$= \phi(z_0) \int_0^{2\pi} \frac{1}{Re^{i\varphi}} iRe^{i\varphi}\, d\varphi + I$$

$$= 2\pi i\, \phi(z_0) + I$$

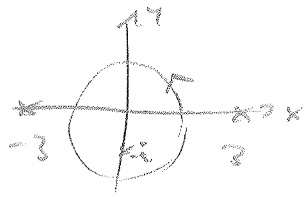
where

$$|I| \le \max_{z \in C_0} \left| \frac{\phi(z) - \phi(z_0)}{z - z_0} \right| \operatorname{length}(C_0) < \frac{\varepsilon}{\cancel{R}} 2\pi \cancel{R} = 2\pi\varepsilon$$

(for all $\varepsilon > 0$) $\rightsquigarrow I = 0$ $\square$

<u>Example</u> Let C be the positive oriented circle $|z|=2$.

$$\int_C \frac{z}{(9-z^2)(z+i)}\,dz = \int_C \frac{\phi(z)}{z+i}\,dz$$



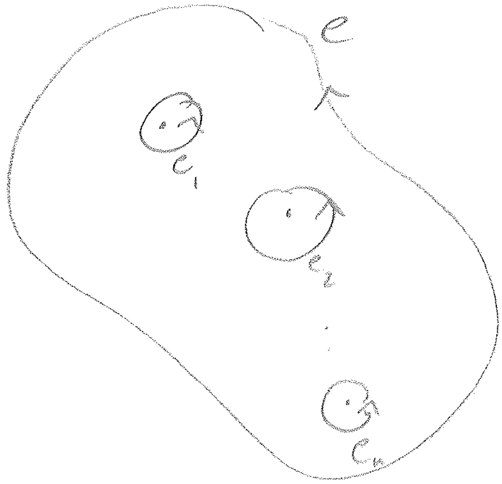
$$\left(\text{where } \phi(z) = \frac{z}{9-z^2} \text{ holomorphic on and inside } C\right) = 2\pi i\, \phi(-i)$$

$$= 2\pi i \frac{-i}{9-(-i)^2} = \frac{\pi}{5}$$

<u>Theorem 4.10</u> (The Residue Theorem) Let f be holomorphic on and inside the simple closed contour C, except at a finite numbers of singularities $z_1, \ldots, z_n$, all inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f$$

<u>Proof</u> (in the case $z_1, \ldots z_n$ simple poles or removable singularities, we shall prove the general case later.)



Draw small circles around the singularities $z_1, \ldots, z_n$

By corollary 4.8

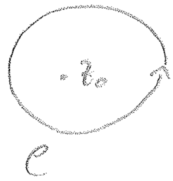
$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

If z_k is removable, then f can be extended to a holomorphic function everywhere inside C_k, so $\int_{C_k} f(z)\,dz = 0$. (and $\operatorname{Res}_{z_k} f(z) = 0$) If z_k is a simple pole, then $f(z) = \frac{\phi(z)}{z-z_k}$ and $\int_{C_k} f(z)\,dz = 2\pi i\, \phi(z_k) = 2\pi i \operatorname{Res}_{z_k} f$. □

<u>Remark:</u> For the general case we shall need the technical results that f has a Laurent series and that
$$\int_C \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \, dz$$
$$= \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n \, dz + \sum_{n=1}^{\infty} b_n \int_C (z-z_0)^{-n} \, dz \,.$$
Then all integrals except $\int_C (z-z_0)^{-1} \, dz = 2\pi i$ vanish. (See Chapter 5).

Why does $\frac{1}{z-z_0}$ have no antiderivative? The problem is that $\log(z-z_0)$ is multi-valued:
$$\log(z-z_0) = w \iff e^{w} = z-z_0 = |z-z_0| \, e^{i \arg(z-z_0)}$$
$$\iff w = \log |z-z_0| + i \arg(z-z_0)$$

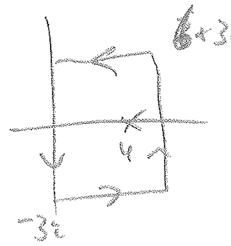
Thus $\log(z-z_0) = \log|z-z_0| + i \arg(z-z_0)$ is only defined up to $2\pi k i$, $k \in \mathbb{Z}$. We can make $\log$ a single-valued function by choosing its principal value, but then we get a discontinuity when we travel once around a circle centred at z_0



$\arg(z-z_0)$ increases by 2π on one circuit of C

Examples applying the residue theorem

1) $$\int_C \frac{z}{(z-4)^2}\,dz$$



$\frac{z}{(z-4)^2}$ has a pole, order 2, at $z=4$, is holomorphic everywhere else

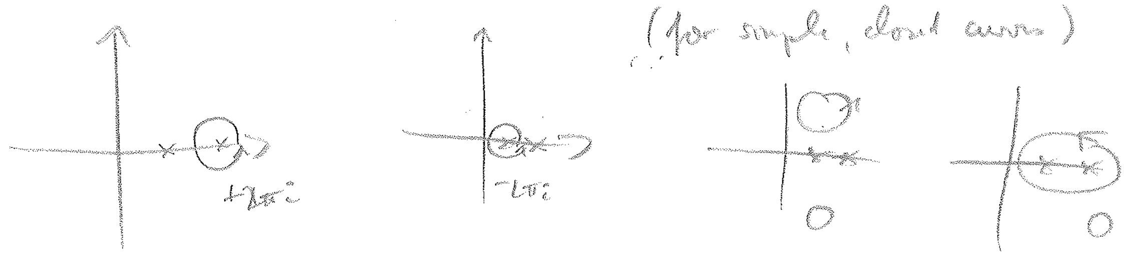
$$\int_C \frac{z}{(z-4)^2}\,dz = 2\pi i \;\mathrm{Res}_4 \frac{z}{(z-4)^2} = 2\pi i \frac{\phi'(4)}{1!} = 2\pi i \qquad \phi(z)=z$$

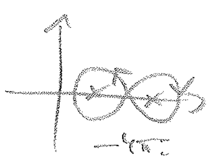
$$\left(\text{or} \quad \frac{z}{(z-4)^2} = \frac{z-4+4}{(z-4)^2} = \frac{4}{(z-4)^2} + \frac{1}{z-4}\right)$$

2) $$\int_C \frac{1}{(1-z)(2-z)}\,dz$$ simple poles at 1 (res $=-1$), 2 (res $=+1$)

$$\left[\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}\right]$$

depending on contour, the integral is 0, $\pm 2\pi i$, ... (for simple, closed curves)



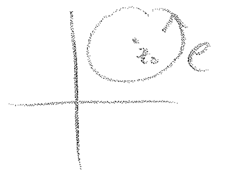


etc

<u>Theorem 4.11</u> If ϕ is holomorphic at z_0 then the derivatives of ϕ of all orders exist and are holomorphic at z_0

<u>Note</u>: There is nothing like this for real functions! Differentiable once $\not\Rightarrow$ differentiable twice!

<u>Sketch of proof</u>: Recall ϕ holomorphic at z_0 means diff'able in some neighbourhood of z_0. By the CIF, $\phi(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\phi(z)}{z-z_0}\, dz$



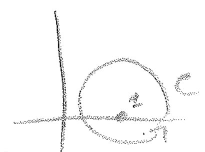
$$\phi'(z_0) = \frac{1}{2\pi i} \frac{d}{dz_0} \int_{\mathcal{C}} \frac{\phi(z)}{z-z_0}\, dz \underset{\substack{\uparrow \\ \text{exchange } \frac{d}{dz_0} \text{ and } \int dz}}{=} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi(z)}{(z-z_0)^2}\, dz$$

analogously, $$\phi^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{\phi(z)}{(z-z_0)^{n+1}}\, dz \qquad \square$$

<u>Corollary 4.12</u> Extended version of Cauchy's Integral Formula

Let ϕ be holomorphic on & inside a simple closed contour $\mathcal{C}$ and z_0 be inside $\mathcal{C}$. Then $\forall n \geq 0$ $$\phi^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{\phi(z)}{(z-z_0)^{n+1}}\, dz$$

<u>Example</u> Compute $\int_{\mathcal{C}} \frac{z^3}{(z-1)^4}\, dz$ where $\mathcal{C}$ winds once around $z=1$



Set $\phi(z) = z^3$

$$\frac{3!}{2\pi i} \int_{\mathcal{C}} \frac{z^3}{(z-1)^4}\, dz = \phi'''(1) = 6 \quad \rightsquigarrow \quad \int_{\mathcal{C}} \frac{z^3}{(z-1)^4}\, dz = 2\pi i$$

or use residue theorem. But to compute $\operatorname{Res}_1 \frac{z^3}{(z-1)^4}$ involves the same calculation.