5 Further applications of contour integration

5.1

In this section (which is really a continuation of section 4) we prove more consequences of Cauchy's Thm, in particular the missing proofs of the theorems about Taylor & Laurent series stated in section 3, and we apply contour integration to the evaluation of <u>real</u> integrals and the summation of <u>real</u> series.

<u>Proposition 5.1</u> (Gauss's mean value theorem)
Let f be holo on & inside a circle C radius R centre z_0. Then $f(z_0)$ is the mean value of f on C; i.e. $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta$

<u>Proof</u> By the C.I.F. $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$

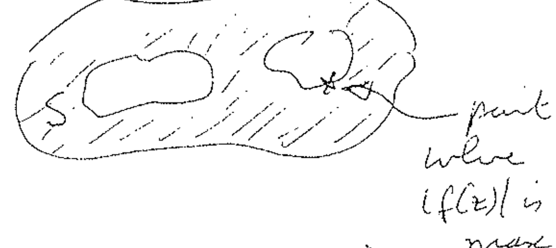
$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{Re^{i\theta}} \cdot iRe^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta \quad \square$$

It follows from 5.1 that <u>either</u>
$|f(z)| > |f(z_0)|$ for some z on C
<u>or</u> $|f(z)| = |f(z_0)|$ $\forall z$ on C, in which case we can show that $f(z)$ has to be constant (details omitted). We deduce

<u>Proposition 5.2</u> (The maximum modulus principle)
If f is holo. and non-constant on a domain $S \subset \mathbb{C}$ then max $\{|f(z)| : z \in S\}$ occurs on the <u>boundary</u> of S.

cf. Murdo

<u>Proof</u> omitted, but follows quite easily from 5.1. $\square$



Now suppose f is entire (holo. on whole of $\mathbb{C}$). As $|z|$ gets larger we can find points with $|f(z)|$ larger & larger (by 5.2). Does this mean $|f(z)|$ must be unbounded for $z \in \mathbb{C}$? We shall prove this soon. First we need an estimate for the size of $|f^{(n)}(z_0)|$:- (generalising Gauss's estimate for $|f'(z_0)|$)

Proposition 5.3 (Cauchy's Inequality (or Estimate))

Let f be holo. on & inside the circle C centre z_0, radius R.
Let $M = \max\{|f(z)| : z \text{ on } C\}$. Then

$$|f^{(n)}(z_0)| \leq \frac{n!\,M}{R^n} \quad \forall n \geq 0$$

Pf From C.I.F. (extended version)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\therefore \; |f^{(n)}(z_0)| \leq \left|\frac{n!}{2\pi i}\right| \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n!\,M}{R^n} \qquad \square$$

Theorem 5.4 (Liouville's Theorem) If f is entire (i.e. holo $\forall z \in \mathbb{C}$) and bounded (i.e. $\exists M$ s.t. $|f(z)| \leq M \;\; \forall z \in \mathbb{C}$) then f is constant.

Pf Take circle centre z_0 radius R. Then $|f'(z_0)| \leq \frac{M}{R}$ (by 5.3) $\forall z_0$
True $\forall R$ $\therefore$ $|f'(z_0)| = 0$ $\forall z_0$ $\therefore$ $f'(z_0) = 0$, $\therefore$ $f = u + iv$ has u & v constant $\therefore$ f is constant //
$\therefore \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0 \;\; \forall z$

Note This is false for real functions - E.g. $\sin x$ & $\cos x$ are bounded on $\mathbb{R}$ but are not constant!

Theorem 5.5 (The Fundamental Theorem of Algebra)
Any polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ $\left\{\begin{matrix} a_j \in \mathbb{C} \; \forall j \\ a_n \neq 0, n \geq 1 \end{matrix}\right\}$ has at least one root in $\mathbb{C}$.

Proof If not, let $f(z) = 1/P(z)$ & $f(z)$ will be entire. We shall show that $f(z)$ is bdd, hence constant (Liouville) which contradicts $n \geq 1$.

$$P(z) = z^n\Bigg(\underbrace{\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}\right)}_{\text{call this } w} + a_n\Bigg) = z^n(a_n + w)$$

Choose R s.t. for $|z| > R$ the terms $\left|\frac{a_0}{z^n}\right|, \ldots, \left|\frac{a_{n-1}}{z}\right|$ are all $< \frac{|a_n|}{2n}$

Then for $|z| > R$, $|w| < \frac{|a_n|}{2}$ so $|a_n + w| > \frac{|a_n|}{2}$.

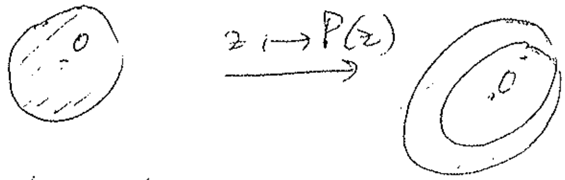
Hence for $|z| > R$, $|f(z)| < \frac{2}{|a_n| R^n}$.

Thus if M = max. value $|f(z)|$ for $\{z : |z| \le R\}$ then $|f(z)| \le \max\left\{M, \dfrac{2}{|a_n|R^n}\right\}$ $\forall z \in \mathbb{C}$. So f is bounded. □

<u>Notes:</u> 1) Could use Max Mod Princ. to show $|f(z)| \le \dfrac{2}{|a_n|R^n}$ $\forall z \in \mathbb{C}$, not just for $|z| > R$.

2) By applying remainder thm it follows from (5.5) that every complex poly. of degree n factors into n linear terms.

3) There are alternative proofs of (5.5) using <u>topology</u>



(boundary of large disc wraps n times around 0 in image)

But it is interesting to see that we need <u>analysis</u> or <u>topology</u> to prove what is essentially an <u>algebraic</u> result.

<u>Counting zeros of holomorphic functions</u>

<u>Prop. 5.6</u>

(i) If z_0 is a zero of f of multiplicity m then z_0 is a simple pole of $\dfrac{f'}{f}$ with residue m

(ii) If f is holo. on and inside a simple closed contour C and has a finite no. of zeros inside C, and none on C, then:

No. of zeros of f inside C (counted with multiplicity) $$N_f = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

Proof (i) $f(z) = (z-z_0)^m g(z)$ with g holo. & $g(z_0) \neq 0$

$$\therefore f'(z) = m(z-z_0)^{m-1} g(z) + (z-z_0)^m g'(z)$$

$$\therefore \frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{holo. at } z_0}$$

$\therefore$ z_0 is a simple pole of $\dfrac{f'}{f}$, residue m

(ii) follows by applying the residue theorem to $\dfrac{f'}{f}$ □

NOT COVERED

Theorem 5.7 (Rouché's Theorem)

If f & g are holo. on & inside a simple closed contour C, & $|f(z)| > |g(z)|$ $\forall z$ on C then $N_{f+g} = N_f$ (where N_f denotes the no. of zeros of f inside C, counted with multiplicity)

Proof (sketch) Let $\Phi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z)+tg'(z)}{f(z)+tg(z)}\,dz \quad 0\le t \le 1$

So $\Phi(0) = N_f$ & $\Phi(1) = N_{f+g}$

$\forall z$ on C $|f(z)+tg(z)| \geq \big|\,|f(z)| - t|g(z)|\,\big| > 0$ (since $|f(z)|>|g(z)|$)

So $f+tg$ has no zeros on C

Hence by 5.6(ii) $\Phi(t) = N_{f+tg}$ (for each t, $0\le t\le 1$)

But Φ is a continuous function of t (as a small change in t makes a small change in $\int_C \frac{f'+tg'}{f+tg}$). As $\Phi(t)$ is an integer we deduce Φ is constant □

Note Rouché's Thm gives another proof of the Fund. Thm. of Alg. On a large circle we know that $|a_n z^n| > |a_0 + a_1 z + \dots a_{n-1}z^{n-1}|$ so by 5.7 $a_0 + a_1 z + \dots + a_n z^n$ has the same no. of zeros as $a_n z^n$.

Examples applying Rouché's Theorem

(1) Determine the no. of roots of $z^7 - 6z^3 + z - 1$ inside the circle $|z|=1$

Solution Put $f(z) = -6z^3 \quad g(z) = z^7 + z - 1$

Then $\forall z$ on $|z|=1$, $|f(z)| = 6$ & $|g(z)| \le 3$

So, by Rouché, $N_{f+g} = N_f = 3$ i.e. $f+g$ has 3 roots inside $|z|=1$ (counted with multiplicity)

(ii) Determine the no. of roots of $2z^5-6z^2+z+1$ in the annulus $1 \le |z| \le 2$

<u>Solution</u> Put $f(z) = -6z^2$ $\quad g(z) = 2z^5+z+1$

Then for $|z|=1$, $\quad |f(z)|=6$ & $|g(z)| \le 4$

So $f+g$ has 2 roots inside $|z|=1$

Next put $f(z) = 2z^5$ $\quad g(z) = -6z^2+z+1$

Then for $|z|=2$, $\quad |f(z)| = 64$ & $|g(z)| \le 24+2+1 = 27$

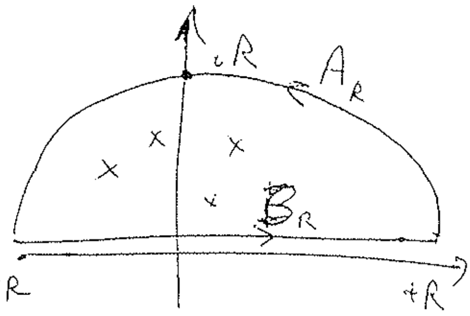
So $f+g$ has 5 roots inside $|z|=2$

$\therefore$ No. of roots in the annulus is $5-2=3$

End of material for test ??

Application of contour integration to real integrals

f holo on $\mathbb{R}$ & only isolated singularities on upper half-plane.



closed contour $C_R = A_R + B_R$

By residue thm $\displaystyle\int_{C_R} f = 2\pi i \left(\sum \text{residues inside } C_R\right)$

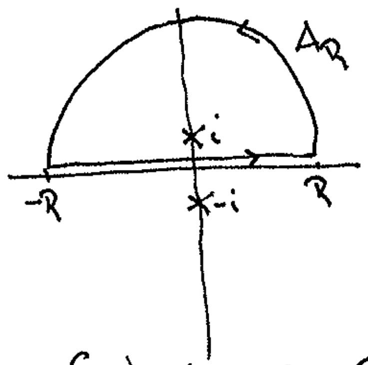
If we have $\displaystyle\int_{A_R} f$ tending to 0 as $R \to \infty$ we can deduce the value of $\displaystyle\lim_{R\to\infty}\int_{B_R} f(z)\,dz$ i.e. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

<u>Variations</u> (i) If f is <u>even</u> (i.e. $f(-x)=f(x)$ $\forall x \in \mathbb{R}$) we can also find $\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$

(ii) If f has singularities on $\mathbb{R}$ we may still be able to compute $\int^\infty f(x)\,dx$ by using a suitable contour 

Example Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$.

$f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)}$ has singularities at i and $-i$.



$(R>1)$

So $\displaystyle\int_{-R}^{R} \frac{1}{x^2+1}\,dx + \int_{A_R} \frac{1}{z^2+1}\,dz = 2\pi i \,\mathrm{Res}(f,i) = 2\pi i \cdot \frac{1}{i+i} = \pi$.

To estimate $\displaystyle\int_{A_R} \frac{1}{z^2+1}\,dz$ note that on A_R we have $|z^2+1| \geq ||z|^2-1| = R^2-1$

so $\left|\dfrac{1}{z^2+1}\right| \leq \dfrac{1}{R^2-1}$. Moreover, length of A_R is πR.

Thus: $\displaystyle\left|\int_{A_R} \frac{1}{z^2+1}\,dz\right| \leq \frac{\pi R}{R^2-1}$ which tends to 0 as R tends to ∞.

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$.

Check: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \left[\tan^{-1}x\right]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$. ✓

Rather than do an example where there is a singularity on the real axis, we conclude with an example showing how contour integrals can be used to sum series:

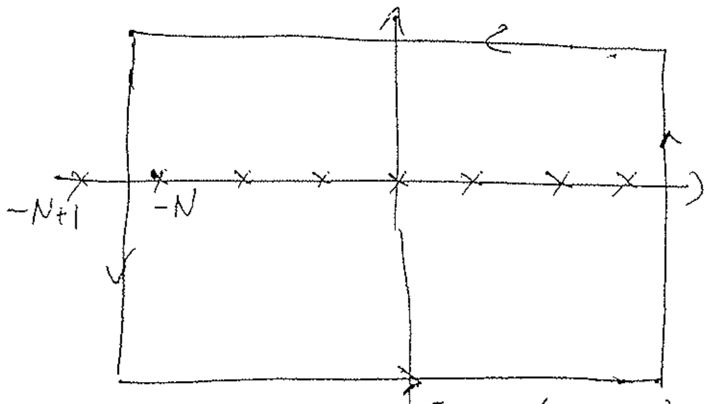
Example 2 To prove $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(not for exam: sketch only)

Let $f(z) = \frac{1}{z^2} \cot \pi z$.

At each $n \neq 0$ $f(z)$ has a simple pole, residue $\frac{1}{n^2 \pi}$ (see overleaf)

At $n = 0$ $f(z)$ has a triple pole, residue $-\frac{\pi}{3}$ (" ")



C_N (square)

$$\int_{C_N} f(z)\,dz = \sum_{n=-N}^{n=+N} 2\pi i \operatorname{Res}(f, n)$$

But can show (by estimating $\max|f(z)|$ on each side of C_N) that $\int_{C_N} f(z)\,dz$ tends to 0 as N tends to ∞

Hence $\sum_{-\infty}^{\infty} 2\pi i \operatorname{Res}(f,n) = 0 \quad \therefore \sum_{-\infty}^{\infty} \operatorname{Res}(f,n) = 0$

$$\therefore -\frac{\pi}{3} + 2\sum_{1}^{\infty} \frac{1}{n^2\pi} = 0 \quad \therefore \boxed{\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

(first proved by Euler by other method)

Notes
1) Can sum other series in similar ways.
2) In any exam question you would be given the f & C_N to use.
3) To compute residues in the example above: $\sin(\pi z - \pi n + n\pi) = \sin(\pi z - \pi n)\cos n\pi = \pi(z-n)\cos n\pi$

$z = n$ (simple pole): Residue $= \frac{1}{n^2} \frac{\cos \pi n}{\pi \cos \pi n} = \frac{1}{n^2 \pi}$

$z = 0$ (triple pole): $\frac{1}{z^2} \cdot \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z^2} \frac{\left(1 - \frac{(\pi z)^2}{2!} + \dots\right)}{\left(\pi z - \frac{(\pi z)^3}{3!} + \dots\right)} = \frac{1}{\pi z^3}\left(1 - \frac{(\pi z)^2}{2!} + \dots\right)\left(1 + \frac{(\pi z)^2}{3!} - \dots\right)$

$\therefore$ coeff of $\frac{1}{z} = \frac{1}{\pi}\left(\frac{\pi^2}{6} - \frac{\pi^2}{2}\right)$