5 Further applications of contour integration

This section contains more consequences of Cauchy's Theorem, gives missing proofs about Taylor and Laurent series (see section 3) and applies contour integration to the evaluation of real integrals and the summation of real sums.

3.12 b

Proposition 5.1 (Gauß mean value theorem)

Let f be holomorphic on and inside a circle C with radius R and centre z_0. Then $f(z_0)$ is the mean value of f on C, i.e.

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta$$

Proof: By CIF, $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it})}{\cancel{z_0} + Re^{it} - \cancel{z_0}}\, iRe^{it}\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\, dt$$

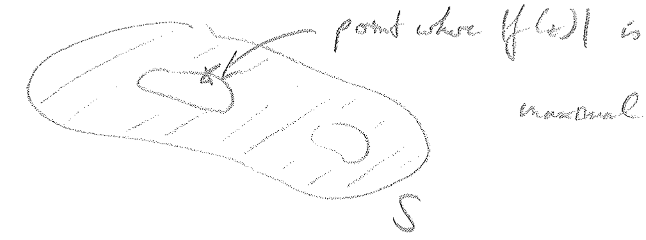
It follows from 5.1 that either $|f(z)| > |f(z_0)|$ for some $z \in C$ or $|f(z)| = |f(z_0)|$ $\forall z \in C$, in which case one can show that $f(z)$ has to be constant (details omitted).

We deduce

Proposition 5.2 (The maximum modulus principle)

If f is holomorphic and non-constant on a domain $S \subset \mathbb{C}$ then $\max\{|f(z)| : z \in S\}$ occurs on the boundary of S

Proof: follows from 5.1, omitted.



Now suppose f is entire (holomorphic on all of $\mathbb{C}$). As $|z|$ gets larger we can find points with $|f(z)|$ larger and larger (by 5.2). Does this mean $|f(z)|$ must be unbounded for $z \in \mathbb{C}$? To prove this, we need an estimate for the size of $|f^{(n)}(z_0)|$:

Proposition 5.3 (Cauchy's Inequality (or estimate))

Let f be holomorphic on and inside the circle C with radius R and centre z_0. Let $M = \max\{|f(z)| : z \in C\}$. Then

$$|f^{(n)}(z_0)| \leq n! \frac{M}{R^n} \quad \forall n \geq 0$$

Proof: By CIF,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\leadsto \left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R = \frac{n!\,M}{R^n}$$

□

<u>Theorem 5.4</u> (Liouville's Theorem) If f is <u>entire</u> (i.e. holomorphic on all of $\mathbb{C}$) and bounded (i.e. $\exists M$ s.t. $|f(z)| \leq M \; \forall z \in \mathbb{C}$) then f is constant

<u>Proof</u>: Take circle with radius R and centre z_0. Then

$$|f'(z_0)| \leq \frac{M}{R} \qquad (5.3 \text{ with } n=1)$$

R arbitrary $\Rightarrow$ $f'(z_0) = 0 \;\; \forall z_0$ $\rightsquigarrow$ f constant

<u>Note</u>: This is false for functions on $\mathbb{R}$: $\sin x$ is bounded but not constant on $\mathbb{R}$. On the other hand, 5.4 implies immediately that $\sin z$ is unbounded on $\mathbb{C}$

<u>Theorem 5.5</u> (The fundamental Theorem of Algebra)

Any polynomial $P(z) = a_0 + a_1 z + \dots + a_n z^n$ with $a_j \in \mathbb{C}$, $a_n \neq 0$, $n \geq 1$ has at least one root in $\mathbb{C}$

<u>Proof</u>: If not, then $f(z) = \dfrac{1}{p(z)}$ is entire. We shall show that $f(z)$ is bounded, hence constant (Liouville) which contradicts $n \geq 1$

$$P(z) = z^n \Bigg(\underbrace{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z}}_{w(z)} + a_n \Bigg) = z^n \left(a_n + w(z) \right)$$

Choose R such that $|w(z)| < \frac{1}{2}|a_n|$ for $|z| > R$

Thus, for $|z| > R$,

$$|a_n + \omega(z)| > \tfrac{1}{2}|a_n|$$

and hence
$$|f(z)| = \frac{1}{|z^n(a_n + \omega(z))|} < \frac{1}{R^n}\,\frac{2}{|a_n|}$$

Thus, if $M = \max\{|f(z)| : |z| \le R\}$ then

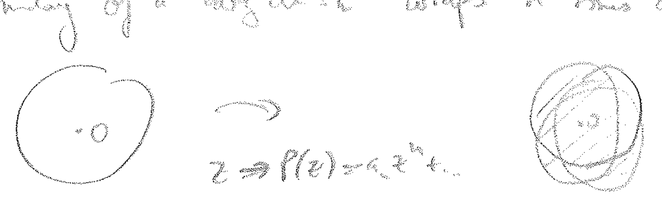
$$|f(z)| \le \max\left\{M, \frac{2}{|a_n| R^n}\right\} \quad \forall z \in \mathbb{C}. \text{ So } f \text{ is bounded } \square$$

<u>Notes</u> (i) Could use Max Mod Principle to show $|f(z)| < \dfrac{2}{|a_n| R^n}$ $\forall z \in \mathbb{C}$, not just for $|z| > R$

(ii) By applying (5.5) iteratively it follows that <u>every complex polynomial of degree $n \ge 1$ factors into n linear terms</u>

$a_n(z - z_1)(z - z_2) \dots (z - z_n)$ i.e. n roots

(iii) There are alternative proofs of (5.5) using <u>topology</u> (the boundary of a large disk wraps n times around 0 in image)



but essentially we need <u>analysis</u> or <u>topology</u> to prove an <u>algebraic</u> result.

Counting zeros of holomorphic functions

6.12

Proposition 5.6

(i) If z_0 is a zero of f of multiplicity m then z_0 is a simple pole of f'/f with residue m

(ii) If f is holomorphic on and inside a simple closed contour C and has a finite number of zeros inside C and none on C then

Number of zeros inside C (counted with multiplicity) $\quad N_f = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz$

Proof: (i) $f(z) = (z-z_0)^m g(z)$ with g holo, $g(z_0) \neq 0$

$\leadsto f'(z) = m\,(z-z_0)^{m-1} g(z) + (z-z_0)^m g'(z)$

$$\leadsto \frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{holo at } z_0}$$

$\leadsto z_0$ is a simple pole of $\frac{f'}{f}$ with residue m

(ii) follows by applying the residue theorem to f'/f. $\square$

Theorem 5.7 (Rouché's Theorem)

If f & g are holomorphic on and inside a simple closed contour C

and $|f(z)| > |g(z)|$ $\forall z$ on C then $N_{f+g} = N_f$

(where N_f denotes the number of f inside C, counted with multiplicity)

Proof (Sketch) Let $\Phi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + t g'(z)}{f(z) + t g(z)} \, dz$, $0 \le t \le 1$

By (5.6)(ii), $\Phi(t) = N_{f+tg}$, provided $f+tg$ has no zeros on C.

This is true, as $\forall z \in C$ $|f(z) + t g(z)| \geq \big| |f(z)| - t|g(z)| \big| > 0$ by assumpt.

Now $\Phi(t)$ is a continuous function, $\Phi(0) = N_f$, $\Phi(1) = N_{f+g}$,

and $\Phi(t)$ can only assume integer values $\rightsquigarrow$ $N_{f+g} = N_f$ ... $\Box$

Note: Rouché's Theorem gives another proof of the fundamental theorem of algebra:

On a large circle we know that $|a_n z^n| > |a_0 + a_1 z + \dots + a_{n-1} z^{n-1}|$, so

by 5.7. $a_0 + a_1 z + \dots + a_n z^n$ has the same number of zeros as $a_n z^n$.

Examples applying Rouché's Theorem:

1) Determine the number of roots of $z^7 - 6z^3 + z - 1$ inside the

disk $|z| = 1$.

Solution: $f(z) = -6z^3$ $g(z) = z^7 + z - 1$

for $|z| = 1$ $|f(z)| = 6$, $|g(z)| \leq 3$, $N_{f+g} = N_f = 3$

2) Determine the number of roots of $2z^5-6z^2+z+1$

in the annulus $1 \le z \le 2$:

Solution: a) $f(z) = -6z^2$, $g(z) = 2z^5+z+1$

for $|z|=1$ $|f(z)|=6$, $|g(z)| \le 4$, $N_{f+g}=2$

b) $f(z) = 2z^5$, $g(z) = -6z^2+z+1$

for $|z|=2$ $|f(z)|=64$, $|g(z)| \le 24+2+1=27$, $N_{f+g}=5$

$f+g$ has 5 roots inside $|z|=2$, two of these inside $|z|=1$

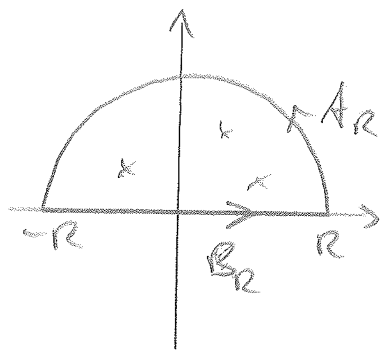
$\Rightarrow$ Number of roots in annulus is $5-2=3$

10.12.(14)

<u>Application of contour integration to real integrals</u>

f holomorphic on $\mathbb{R}$ & only isolated singularities

on the upper half plane



choose closed contour $C_R = A_R + B_R$

By residue theorem $\displaystyle\int_{C_R} f(z)\,dz = 2\pi i \sum (\text{residues inside } C_R)$

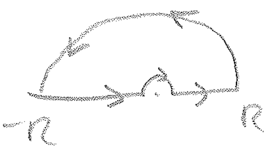
If we have $\displaystyle\lim_{R\to\infty}\int_{A_R} f(z)\,dz = 0$ then we can compute the value

of $\displaystyle\lim_{R\to\infty}\int_{B_R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$ via "sum of residues"!

__Variations__ (i) If f is even (i.e. $f(-x) = f(x)\ \forall x \in \mathbb{R}$)

we can find $\displaystyle\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$

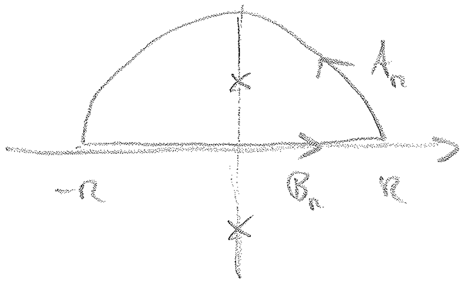
(ii) If f has singularities on $\mathbb{R}$ we may still be able to compute $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ by using a suitable contour



__Example 1__ Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ and $\displaystyle\int_0^{\infty} \frac{1}{x^2+1}\,dx$

$f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)}$ simple poles at $z = \pm i$ $\quad \operatorname{Res}_i f = \dfrac{1}{i+i} = -\dfrac{i}{2}$

$\operatorname{Res}_{-i} f = \dfrac{1}{-i-i} = +\dfrac{i}{2}$



for $R > 1$ we have

$$\int_{A_R} f(z)\,dz + \int_{B_R} f(z)\,dz = \int_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}_i f = \pi$$

$$\left|\int_{A_R} f(z)\,dz\right| \le \frac{1}{R^2-1}\pi R \to 0 \text{ as } R \to \infty$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{R\to\infty}\int_{B_R} f(z)\,dz = \pi$

and $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$

Of course, we could have used

$$\int \frac{dx}{x^2+1} = \arctan x \quad \text{to get} \quad \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \arctan x \Big|_{-\infty}^{\infty} = \pi$$

Example 2 Find $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$ and $\displaystyle\int_0^{\infty} \frac{\cos x}{x^2+1}\,dx$

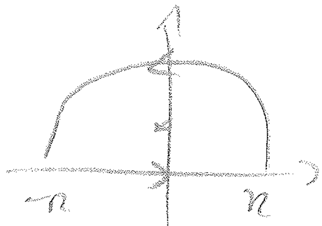
$$f(z) = \frac{\cos z}{z^2+1} = \frac{\cos z}{(z+i)(z-i)} \quad \text{simple pole at } z = \pm i$$

but problems with $\left|\int_{A_R}\right|$ as $\cos z = \frac{1}{2}(e^{iz}+e^{-iz})$ grows too fast!

use $f(z) = \dfrac{e^{iz}}{z^2+1}$ and $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx$

$\left(\text{as } \displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx = 0\right)$

$$f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)} \quad \text{simple pole at } z = \pm i$$



$$\mathrm{Res}_i f = \frac{e^{i\,i}}{i+i} = -\frac{i}{2e}$$

$$\left(\mathrm{Res}_{-i} f = \frac{e^{i(-i)}}{-i-i} = \frac{ie}{2}\right)$$

$R>1$: $\displaystyle\int_{C_R} f(z)\,dz = 2\pi i\,\mathrm{Res}_i f = \frac{\pi}{e}$

$$\left|\int_{A_R} f(z)\,dz\right| \leq \frac{1}{R^2-1}\pi R \to 0 \text{ as } R \to \infty$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}$, $\displaystyle\int_0^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{2e}$

And this we could not have calculated using "conventional" methods!

(just try MAPLE ...)

If since
$$\max_{z\in A_R} |e^{iz}| = \max_{0\le t\le\pi} |e^{iRe^{it}}|$$

$$= \max_{0\le t\le\pi} |e^{iR(\cos t + i\sin t)}| = \max_{0\le t\le\pi} e^{-R\sin t} = 1$$

important that $0\le t\le\pi$, this would not work in lower half plane!

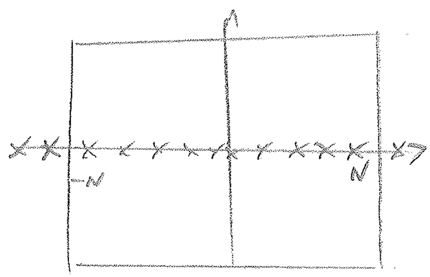
(10.12.5)

Note: The answer for a real integral must be real. A complex answer would indicate an error in the calculation.

Example 3 shows how to use $\int_{\mathcal{C}} f\,dz$ to calculate the sum of a series:

3) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Sketch)

Use that $\cot \pi z$ has simple poles at $n\in\mathbb{Z}$ with residue $\frac{1}{\pi}$



$\mathcal{C}_N$

Square

$n\in\mathbb{Z}\setminus\{0\}$:

$$\operatorname{Res}_n \frac{1}{z^2}\cot\pi z = \frac{1}{n^2\pi}$$

$$\frac{1}{z^2}\cot\pi z = \frac{1}{z^2}\left(\frac{1}{\pi z} - \frac{\pi z}{3} + \dots\right) \rightsquigarrow \operatorname{Res}_0 \frac{1}{z^2}\cot\pi z = -\frac{\pi}{3}$$

Thus, $\int_{C_N} \frac{1}{z^2}\cot \pi z \, dz =$

$$2\pi i \sum_{n=-N}^{N} \operatorname{Res}_{z=n}\left(\frac{1}{z^2}\cot\pi z\right) = 2\pi i\Big(\overset{\overset{n=0}{\downarrow}}{-\frac{\pi}{3}} + 2\sum_{n=1}^{N}\frac{1}{n^2\pi}\Big)$$

But one can show by estimating $\max |f(z)|$ on each side of C_N

that $\int_{C_N} f(z)\,dz$ tends to zero as N tends to ∞.

Hence
$$0 = -\frac{\pi}{3} + 2\sum_{n=1}^{\infty}\frac{1}{n^2\pi},$$

or,
$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

<u>Notes:</u> 1) Can sum other series in similar ways

2) In any exam question you would be given the f & C_N to use

Diff. forms; 75a insert

Example 4)

Compute $I(\lambda) \equiv \int_0^{2\pi} \frac{dt}{1+\lambda \cos t}$, $|\lambda| < 1$

Integral is of the type $\int_0^{2\pi} f(\cos t, \sin t)\, dt$

like evaluation of a contour integral along the unit circle $\mathcal{C}$

$$\gamma(t) = e^{it}, \quad \gamma'(t) = i\, e^{it}, \quad \cos t = \tfrac{1}{2}\left(z+\tfrac{1}{z}\right), \quad \sin t = \tfrac{1}{2i}\left(z-\tfrac{1}{z}\right)$$

$$\int_0^{2\pi} f(\cos t, \sin t)\, dt = \int_0^{2\pi} f(\cos t, \sin t)\, \frac{i e^{it}}{i e^{it}}\, dt$$

$$= \int_{\mathcal{C}} f\left(\tfrac{1}{2}\left(z+\tfrac{1}{z}\right), \tfrac{1}{2i}\left(z+\tfrac{1}{z}\right)\right) \frac{dz}{iz}$$

can be evaluated via residues within the unit circle.

expand in class!

$$I(\lambda) = \int_{\mathcal{C}} \frac{1}{1+\frac{\lambda}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz} = -\frac{2i}{\lambda} \int_{\mathcal{C}} \frac{dz}{z^2 + \frac{2}{\lambda} z + 1}$$

simple poles at $z_\pm = -\frac{1}{\lambda} \pm \frac{1}{\lambda}\sqrt{1-\lambda^2}$ $\quad |z_+| < 1, \quad |z_-| > 1$

$$I(\lambda) = -\frac{2i}{\lambda}\, 2\pi i \, \operatorname{Res}_{z=z_+} \frac{1}{(z-z_+)(z-z_-)} = \frac{4\pi}{\lambda} \frac{1}{z_+ - z_-} = \frac{2\pi}{\sqrt{1-\lambda^2}}, \quad |\lambda| < 1$$

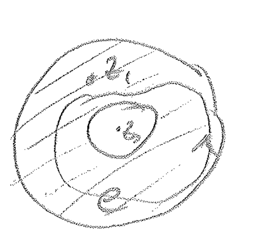
Filling the gaps: the missing proofs

Laurent's Theorem (viz 3.7) Let f be holomorphic on

$A = \{ z : R_1 < |z_1 - z_0| < R_2 \}$ (where R_1 can be 0 and R_2 can be ∞).

Let C be a simple closed positively oriented contour around z_0, in A.

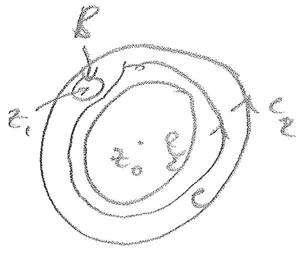
Let z_1 be any point of A. Then



$$f(z_1) = \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n + \sum_{n=1}^{\infty} b_n (z_1 - z_0)^{-n}$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z)}{(z-z_0)^{n+1}}\, dz$ and $b_n = \dfrac{1}{2\pi i} \displaystyle\int_C f(z)(z-z_0)^{n-1}\, dz$

Proof: Take circles C_1, C_2 centred at z_0, both inside A and with z_1 and C lying in between C_1 and C_2. Take B a small circle around z_1 not meeting C_1 or C_2 (may meet C). We have



$$(1) \qquad f(z_1) \overset{\text{CIF}}{=} \frac{1}{2\pi i} \int_B \frac{f(z)}{z - z_1}\, dz \overset{\text{Cauchy Thm}}{=} \int_{C_2} \frac{f(z)}{z - z_1}\, dz - \int_{C_1} \frac{f(z)}{z - z_1}\, dz$$

Let $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z)}{(z-z_0)^{n+1}}\, dz = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \dfrac{f(z)}{(z-z_0)^{n+1}}\, dz = \dfrac{1}{2\pi i} \displaystyle\int_{C_2} \dfrac{f(z)}{(z-z_0)^{n+1}}\, dz$

and b_n similarly.

If we can show that

$$(2) \quad \sum_{n=0}^{\infty}{}' a_n (z_1 - z_0)^n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1}\, dz$$

and

$$(3) \quad \sum_{n=1}^{\infty}{}' b_n (z_1 - z_0)^{-n} = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z_1 - z}\, dz$$

then (2) and (3) together with (1) prove the theorem.

To prove (2), write

$$\sum_{n=0}^{N}{}' a_n (z_1 - z_0)^n = \sum_{n=0}^{N}{}' \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-z_0)^{n+1}}\, dz\, (z_1 - z_0)^n$$

$$= \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_0} \underbrace{\sum_{n=0}^{N}{}' \left(\frac{z_1 - z_0}{z - z_0}\right)^n}_{\frac{1 - \left(\frac{z_1 - z_0}{z - z_0}\right)^{N+1}}{1 - \frac{z_1 - z_0}{z - z_0}}} dz$$

$$= \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1} \left(1 - \left(\frac{z_1 - z_0}{z - z_0}\right)^{N+1}\right) dz$$

now $\left|\dfrac{z_1 - z_0}{z - z_0}\right| \leq \rho < 1$ for $z \in C_2$, and $\left|\dfrac{1}{2\pi i} \displaystyle\int_{C_2} \frac{f(z)}{z - z_1} \left(\frac{z_1 - z_0}{z - z_1}\right)^{N+1} dz\right|$

$$\leq \frac{1}{2\pi}\, 2\pi R_2\, \rho^N \max_{z \in C_2} \left|\frac{f(z)}{z - z_1}\right| \to 0 \quad \text{as } N \to \infty$$

thus $\displaystyle\sum_{n=0}^{\infty}{}' a_n (z_1 - z_0)^n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1}\, dz$ exists, proving (2).

A similar argument holds for (3).

$\square$

Corollary 1 : Taylor's Theorem (viz 3.6) Let f be holomorphic everywhere on $D = \{z : |z-z_0| < R\}$. Then $f^{(n)}(z_0)$ exists $\forall n \geq 0$ and for any z_1 in D

$$f(z_1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z_1 - z_0)^n$$

Proof : Let C be a circle in D centred at z_0 and containing z_1.

By Laurent's Theorem

$$f(z_1) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \, (z_1 - z_0)^n + \sum_{n=1}^{\infty} \frac{\frac{1}{2\pi i} \int_C f(z)(z-z_0)^{n-1} dz}{(z_1 - z_0)^n}$$

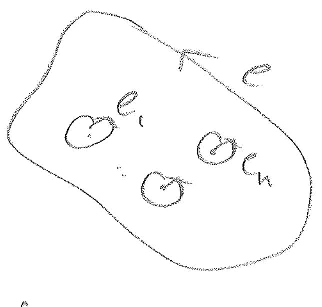
$$= \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z_1 - z_0)^n}_{\text{(extended CIF)}} + \underbrace{\sum_{n=1}^{\infty} 0 \, (z_1 - z_0)^{-n}}_{\text{(Cauchy Theorem)}} \qquad \square$$

Corollary 2 : The Residue Theorem (viz 4.10) (general proof)

Let f be holomorphic on and inside a simple closed contour C except at a finite number of singularities $z_1, \dots, z_n$ inside (but not on) C.

Then, if C is positively oriented,

$$\int_C f(z) dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z_j}(f)$$



Proof By deformation principle $\int_C f(z) dz = \sum_{j=1}^{n} \int_{C_j} f(z) dz$, but this is just 2πi times the coeff. b_1 of the respective Laurent series, i.e. the residue at z_j

$\square$