MTH5105 Differential and Integral Analysis 2008-2009

Midterm Test

Problem 1: Let $f(x) = 1/x$.

(a) Determine the Taylor polynomials $T_{3,1}$ and $T_{4,1}$ of degree 3 and 4 at $a = 1$ for f.

[15 marks]

(b) Using the Lagrange form of the remainder, or otherwise, show that

$$T_{3,1}(x) < f(x) < T_{4,1}(x) \quad \text{for all } x > 1 .$$

[15 marks]

Solution: (a) $f(x) = 1/x$, $f'(x) = -1/x^2$, $f''(x) = 2/x^3$, $f'''(x) = -6/x^4$, $f^{(4)}(x) = 24/x^5$, and therefore $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$, $f'''(1) = -6$, $f^{(4)}(1) = 24$.

[5 marks]

Hence

$$\begin{aligned}
T_{3,1}(x) =& \frac{1}{0!}1 + \frac{(-1)}{1!}(x-1) + \frac{2}{2!}(x-1)^2 + \frac{(-6)}{3!}(x-1)^3 \\
=& 1 - (x-1) + (x-1)^2 - (x-1)^3 , \\
T_{4,1}(x) =& \frac{1}{0!}1 + \frac{(-1)}{1!}(x-1) + \frac{2}{2!}(x-1)^2 + \frac{(-6)}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 \\
=& 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 .
\end{aligned}$$

[5 marks each]

(b) For $x > 1$ there is a $c \in (1, x)$

[5 marks]

such that

$$f(x) = T_{3,1}(x) + \frac{24/c^5}{4!}(x-1)^4 = T_{3,1}(x) + (x-1)^4/c^5 .$$

[5 marks]

But as $c > 1$, we have $(x-1)^4/c^5 < (x-1)^4$, and thus

$$T_{3,1}(x) < 1/x < T_{4,1}(x) .$$

[5 marks]

Problem 2: (a) Give the definition of $f : \mathbb{R} \to \mathbb{R}$ being differentiable at a point $a \in \mathbb{R}$.

[*10 marks*]

(b) Using the definition, determine whether or not

$$f(x) = \begin{cases} \dfrac{x}{1 + \exp(1/x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at $x = 0$. (For this you may wish to consider the left and right derivatives of $f(x)$ at $x = 0$.) Find $f'(0)$, if it exists. [*20 marks*]

Solution: (a) f is differentiable at $a \in \mathbb{R}$ if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. [*10 marks*]

(b) We need to consider

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1}{1 + \exp(1/x)} \ .$$

[*5 marks*]

But as $\lim_{t \to \infty} \exp(-t) = 0$ we have

$$\lim_{x \to 0^+} \frac{1}{1 + \exp(1/x)} = \lim_{t \to \infty} \frac{1}{1 + \exp(t)} = \lim_{t \to \infty} \frac{\exp(-t)}{\exp(-t) + 1} = 0$$

[*5 marks*]

and

$$\lim_{x \to 0^-} \frac{1}{1 + \exp(1/x)} = \lim_{t \to \infty} \frac{1}{1 + \exp(-t)} = 1 \ .$$

[*5 marks*]

As the left and right limits disagree, the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist. Thus f is not differentiable at 0 and $f'(0)$ does not exist.

[*5 marks*]

Problem 3: (a) State the Mean Value Theorem. [*15 marks*]

(b) Show that for all $x, y \in \mathbb{R}$

$$|\sin(y) - \sin(x)| \leq |y - x| \ .$$

[*25 marks*]

You may assume standard properties of trigonometric functions.

Solution: (a) MVT: Let f be continuous on $[a, b]$ and differentiable on (a, b).

[*5 marks*]

Then there is a $c \in (a, b)$

[*5 marks*]

such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \ .$$

[*5 marks*]

(c) Consider three cases: (a) $x < y$, (b) $x = y$, and (c) $x > y$. For $x = y$ the conclusion is true, and (c) follows from (a) by exchanging x and y. So we only need to consider $x < y$.

[*5 marks*]

Let $x < y$ and apply the MVT to $f(x) = \sin(x)$ on $[x, y]$.

[*5 marks*]

There is a $c \in (x, y)$ such that

$$\cos(c) = \frac{\sin(y) - \sin(x)}{y - x} \ .$$

[*5 marks*]

But $|\cos(c)| \leq 1$, so that

$$1 \geq \left| \frac{\sin(y) - \sin(x)}{y - x} \right| \ .$$

[*5 marks*]

Therefore $|\sin(y) - \sin(x)| \leq |y - x|$.

[*5 marks*]