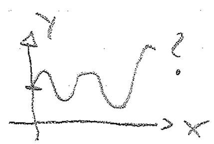
§0 Introduction

What is a differential equation?

$$\underbrace{y' = \frac{dy}{dx}}_{\substack{\text{derivative} \\ \text{"rate of change"}}} = \underbrace{y^2 - x^3}_{\substack{\text{some expression} \\ \text{of } x \text{ and } y}} \qquad y(x) = ?$$



A differential equation relates derivatives with dependent and independent variables.

Why to consider such things?

Because they describe our world, e.g.,

mechanics ($\ddot{\vec{r}} = \vec{F}(\vec{r}, \dot{\vec{r}})$), electronics ($\dot{U} = I/C + \dots$), chemistry ($\dot{c} = -kc$), quantum mechanics ($\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \Delta \psi$), economics ($\dot{S} = \dot{w}$)

They are useful and... nice!

§1 Functions, indefinite integrals, and initial value problems

a) Functions

$y = g(x)$ $\quad$ x: independent variable

y: dependent variable

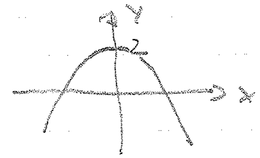
$g(\cdot)$: function

A function is a rule which relates variables from a domain ($x \in D$) with variables of a range ($y \in R$) $\qquad g: D \to R$

$$x \mapsto y = g(x)$$

Example 1: $\quad y = g(x) = -x^2 + 2$

$g(-3) = -(-3)^2 + 2 = -7$ $\qquad$ graph



$g(a) = -a^2 + 2$

Example 2: $\quad x = g(t) = t^3 - t$ $\qquad$ t: independent variable

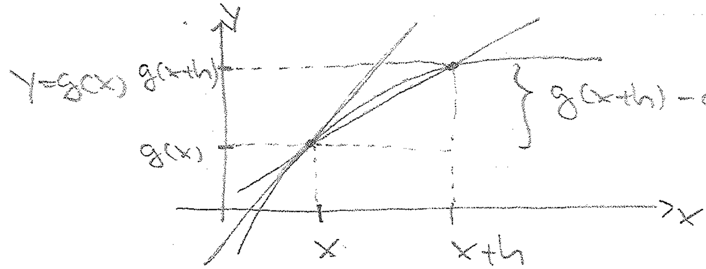
x: dependent variable

$g(1) = 0$

$g(s) = s^3 - s$ $\quad$, $g(1-a) = (1-a)^3 - (1-a)$

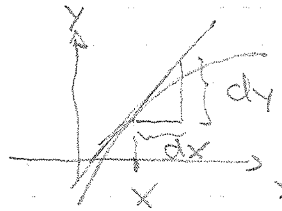
Remark: We will normally use the notation $y = g(x)$ (x: independent, y: dependent). We will often write $y(x)$ or y to denote the function $y = g(x)$.

b) Derivative and indefinite integral.



$$g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = \frac{dg}{dx} = \frac{dy}{dx} = y'$$

(slope of the tangent)



Remark: $g'(x)$ determines the rate of change of the function g at x: $dy = g'(x)\,dx$

Example 1: $g(x) = x^3 \sin(x^2)$

$g'(x) = 3x^2 \sin(x^2) + 2x^4 \cos(x^2)$ (cf. product rule, chain rule, ...)

A function $G(x)$ is called antiderivative of the function $y = g(x)$ if $G'(x) = g(x)$.

Example 2:

$g(x) = x^2$

$G(x) = x^3/3$

$G(x) = x^3/3 - 2$

$G(x) = x^3/3 + C$ (C constant)

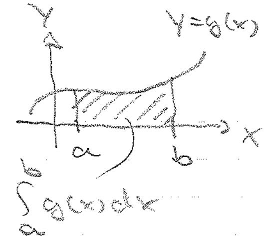
are all antiderivatives

Indefinite integral (synonym for antiderivative, cf. "fundamental theorem")

$$\int g(x)\,dx = G(x) + C$$

Definite integral

$$\int_a^b g(x)\,dx = G(x)\Big|_a^b = G(b) - G(a)$$



Example 3: $y = g(x) = \sqrt{x}$

$G(x) = \int g(x)\,dx = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

$\int_1^2 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^2 = \frac{2}{3}(2^{3/2} - 1) = \frac{2}{3}(2\sqrt{2} - 1)$

c) Initial value problems

A differential equation relates the derivative, e.g. $y' = \frac{dy}{dx}$ with dependent and independent variables, for instance $\frac{dy}{dx} = y^2 - x$ or more general $\frac{dy}{dx} = f(x,y)$.

How to find $y(x)$?

Example 1:

$y' = \frac{dy}{dx} = x^2 + 3$ $\qquad y(x) = ?$

Try integration

$y(x) = \int \frac{dy}{dx}\,dx = \int x^2 + 3\,dx = \frac{1}{3}x^3 + 3x + C$

"general solution"

Check: $\frac{dy}{dx} = x^2 + 3$ ✓

Determine the constant of integration C from a given initial condition

$y(0) = 1/3 \Rightarrow 1/3 = y(0) = 0 + C \Rightarrow C = 1/3$

$y(x) = \frac{1}{3}x^3 + 3x + 1/3$ "Solution of the initial value problem" ("particular solution")

A differential equation, say $\frac{dy}{dx} = f(x,y)$, with initial condition $y(0) = y_0$ is called initial value problem.

Example 2: $\frac{dy}{dx} = x^2 - y^2 \quad y(0) = 3$

Try solution by integration

$$y(x) = \int x^2 - y^2 \, dx = \tfrac{1}{3}x^3 - \underbrace{\int (y(x))^2 dx}_{?\ y(x)\ \text{unknown!}\ ?}$$

Solution by integration does not work if the right hand side depends on y (i.e. on the dependent variable).

$\to$ methods?

§2 First order differential equations

The order of a differential equation is the order of the highest derivative which occurs in the equation

Example 1:

$y' = y^2 + x \qquad (y')^2 + x^2 - y^2 = 1 \qquad$ first order

$y' + x = y\,y'' \qquad$ second order

A differential equation of first order is said to be in normal form if it is written as

$$\frac{dy}{dx} = f(x,y)$$

Example 2:

$y' = y^2 + x$ normal form $\qquad (y')^2 = yx$ not in normal form

A (particular) solution of the first order differential equation $y' = f(x,y)$ is a differentiable function $y = y(x)$ such that $\frac{dy}{dx} = f(x, y(x))$ for all x where $y(x)$ is defined.

a) Solution by substitution

Example 1: Which of the following two functions is a solution of the differential equation

$$y' = -y/(1+x)$$

- $y(x) = 1 + x$

$\frac{dy}{dx} = 1$

$-\frac{y}{1+x} = -\frac{1+x}{1+x} = -1$

$1 \neq -1$

not a solution

- $y(x) = \frac{1}{1+x}$

$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

$-\frac{y}{(1+x)} = -\frac{1}{(1+x)^2}$

$y' = -\frac{1}{(1+x)^2} = -\frac{y}{1+x}$ (for all x !)

$\to$ solution

Example 2: Determine the constant λ such that $y(x) = e^{\lambda x}$ is a solution of $\quad y' = -2y$