$\frac{dy}{dx} = \lambda e^{\lambda x}$, $-2y = -2e^{\lambda x}$, $\lambda e^{\lambda x} = -2e^{\lambda x} \Rightarrow \lambda = -2$

Disadvantage: One has to "guess" a solution

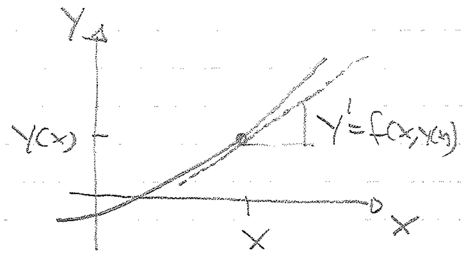
b) Direction field

first order differential equation $y' = f(x,y)$

solution $y = y(x)$

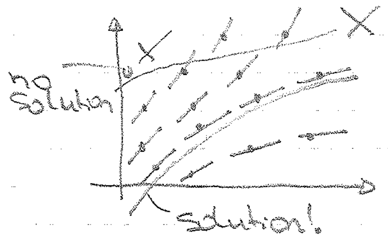
$y' = f(x, y(x))$

$y(x)$ has slope $f(x,y)$ at (x,y)



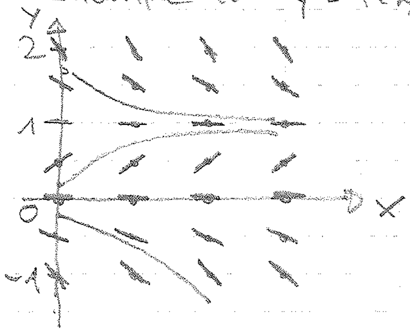
$f(x,y)$ determines the slope of the solution at (x,y)

direction field



draw a slanted line segment with slope $f(x,y)$ at "each" point (x,y)
$\to$ direction field
A solution of the diff. eq. $y' = f(x,y)$ is a curve which is in each point tangential to the direction field

Example 1: $y' = f(x,y) = y(1-y)$



- $y(0) = 0 \Rightarrow y(x) = 0$
- $y(0) = 1 \Rightarrow y(x) = 1$
- $y(0) > 0 \Rightarrow y(x) \to 1 \; (x \to \infty)$
- $y(0) < 0 \Rightarrow y(x) \to -\infty \; (x \to \infty)$

method yields qualitative features of solutions

c) Separation of variables

Example 1: $\frac{dy}{dx} = -2y \quad y(0) = 3$

"separate variables": (treat $\frac{dy}{dx}$ as a fraction) $\frac{dy}{y} = -2dx$

Integrate: $\int \frac{dy}{y} = \int -2dx \Rightarrow \ln|y| + C_1 = -2x + C_2$

$\Rightarrow \ln|y| = -2x + C \quad (C = C_2 - C_1)$ (general solution, implicit form)

Solve for y:

$|y| = e^{C} e^{-2x} \Rightarrow y = e^{C} e^{-2x}$ or $y = -e^{C} e^{-2x}$

$\Rightarrow y = De^{-2x}$ (general solution, explicit form)

Determine constant of integration by initial condition

$y(0) = 3 \Rightarrow 3 = De^{0} \Rightarrow D = 3 \Rightarrow y = 3e^{-2x}$

(solution of initial value problem)

Example 2: $\frac{dy}{dx} = x \cdot y^2 \quad y(0) = -1$

$\frac{dy}{y^2} = x\,dx \Rightarrow \int \frac{dy}{y^2} = \int x\,dx \Rightarrow -\frac{1}{y} + C_1 = \frac{1}{2}x^2 + C_2 \Rightarrow -\frac{1}{y} = \frac{1}{2}x^2 + C$

$y(0) = -1 \Rightarrow -\frac{1}{-1} = 0 + C \Rightarrow C = 1$

$\Rightarrow y = -\frac{1}{1 + \frac{1}{2}x^2} = \frac{-2}{2 + x^2}$

Why and when does separation of variables work?

A first order differential equation $y' = f(x,y)$ is separable if the right hand side is a product of a function depending (solely) on x and a function depending (solely) on y

$$y' = g(x)\,h(y)$$

If $y(x)$ denotes the solution, i.e. $y'(x) = g(x)h(y(x))$ then

$$\frac{1}{h(y(x))}y'(x) = g(x) \Rightarrow \int \frac{1}{h(y(x))}y'(x)\,dx = \int g(x)\,dx$$

Substitution $y = y(x) \Rightarrow dy = y'(x)dx$ yields

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Remark: The integrals may be difficult to solve and it may be difficult as well to obtain the explicit solution.

Example 3: $\frac{dy}{dx} = \frac{2xy}{1+y}$ $\quad y(0) = 2$

$$\Rightarrow \frac{1+y}{y}dy = 2x\,dx \Rightarrow \int\left(\frac{1}{y}+1\right)dy = \int 2x\,dx$$

$\Rightarrow \ln|y| + y = x^2 + C_1$ (general solution, implicit form)

$y(0) = 2 \Rightarrow \ln 2 + 2 = 0 + C_1$

$\Rightarrow \ln y + y = x^2 + 2 + \ln 2$ (implicitely defined solution)

since $y \geq 0$!

$\Rightarrow$ explicit form not easy to compute!

d) First order linear differential equations

A differential equation of the form

$$y' = a(x)\,y$$

is called (first order linear) homogeneous diff. eq.

A differential equation of the form

$$y' = a(x)\,y + b(x)$$

is called (first order linear) inhomogeneous diff. eq.

Example 1:

$y' = \sin(x)\,y$	homogeneous
$y' = e^x y + x$	inhomogeneous
$y' = 1 - y^2 + x$	nonlinear

- Solution of homogeneous equations by separation of variables:

$$y' = a(x)y \Rightarrow \frac{dy}{y} = a(x)dx \Rightarrow \int\frac{dy}{y} = \underbrace{\int a(x)\,dx}_{A(x)}$$

$$\Rightarrow \ln|y| = A(x) + C_1 \Rightarrow |y| = e^{C_1}e^{A(x)}$$

$\Rightarrow y = De^{A(x)} = De^{\int a(x)dx}$ (general solution)

Example 2: $y' = \sin(x)\,y$ $\quad y(0) = 2$

$$\int \frac{dy}{y} = \int \sin(x)dx \Rightarrow \ln|y| = -\cos(x) + C_1'$$

$$\Rightarrow |y| = e^{c} e^{-\cos(x)} \Rightarrow y = De^{-\cos(x)}$$

$$y(0) = 2 \Rightarrow 2 = De^{-1} \Rightarrow D = 2e \Rightarrow y = 2e^{1-\cos(x)}$$

- Solution of inhomogeneous equations by integrating factor method

Example 3: $y' = y + e^{-x}$

$$\Rightarrow \frac{dy}{dx} - y = e^{-x}$$ direct integration impossible

Idea: try to find a function $u(x)$ such that

$$u(x)\frac{dy}{dx} - u(x)y = \frac{d(u(x)y)}{dx} \qquad (*)$$

Then

$$\frac{d(u(x)y)}{dx} = u(x)\left(\frac{dy}{dx} - y\right) = u(x)e^{-x}$$

$$\Rightarrow u(x)y = \int u(x)e^{-x}dx + C_1' \qquad (!)$$

$$\Rightarrow y = \frac{1}{u(x)}C_1' + \frac{1}{u(x)}\int u(x)\, e^{-x}dx$$

$u(x)$ determined by $(*)$:

$$\cancel{u(x)\frac{dy}{dx}} - u(x)y = \frac{du}{dx}y + \cancel{u(x)\frac{dy}{dx}}$$

$$\Rightarrow \frac{du}{dx} = -u(x) \quad \text{(homogeneous differential equation)}$$

$$\Rightarrow u(x) = e^{-x} \quad \text{("integrating factor")}$$

Thus

$$y = \frac{1}{e^{-x}}C_1' + \frac{1}{e^{-x}}\overbrace{\int e^{-x}e^{-x}dx}^{-\frac{1}{2}e^{-2x}}$$

$$= C_1'e^{x} - \tfrac{1}{2}e^{-x} \qquad \text{(general solution)}$$

General case: $\frac{dy}{dx} = a(x)y + b(x)$

$$\Rightarrow \frac{dy}{dx} - a(x)y = b(x)$$

Use the integrating factor $u(x) = e^{-A(x)} = e^{-\int a(x)dx}$

$$\Rightarrow \underbrace{e^{-A(x)}\frac{dy}{dx} - a(x)e^{-A(x)}y}_{\frac{d\, e^{-A(x)}y}{dx}} = e^{-A(x)}b(x) \qquad \text{(product rule!)}$$

$$\Rightarrow e^{-A(x)}y = \int e^{-A(x)}b(x)dx + C_1'$$

$$\Rightarrow y = \underbrace{C_1'e^{A(x)}}_{\text{(hom. sol.)}} + \underbrace{e^{A(x)}\int e^{-A(x)}b(x)dx}_{\text{(part. sol. inhom. eq.)}} \qquad \text{(general solution)}$$

Example 4: $y' = \underbrace{\sin(x)}_{a(x)}y + \underbrace{2x\, e^{-\cos(x)}}_{b(x)}$

$$A(x) = \int a(x)dx = -\cos x$$

$$u(x) = e^{-A(x)} = e^{\cos x} \qquad y' - \sin(x)y = 2xe^{-\cos x}$$

$$\Rightarrow \underbrace{e^{\cos(x)}y' - \sin(x)e^{\cos(x)}y}_{\frac{d\, e^{\cos(x)}y}{dx}} = 2xe^{-\cos x}e^{\cos x}$$