$e^{\cos x} y = \int 2x\,dx = x^2 + C_1' \Rightarrow y = C_1' e^{-\cos x} + x^2 e^{-\cos x}$

- Solution of inhomogeneous equations by variation of parameters (alternative method)

Example 5: $y' = \sin(x) y + 2x\, e^{-\cos x}$

Solve the homogeneous equation

$y' = \sin(x) y \Rightarrow \frac{dy}{y} = \sin(x) \Rightarrow \ln|y| = \cos(x) + C_1'$

$\Rightarrow y = D e^{-\cos(x)}$

"variation of parameters": replace the constant of integration, D, by an (unknown) x-dependent function $\alpha(x)$

$$y = \alpha(x)\, e^{-\cos x} \qquad (*)$$

and determine $\alpha(x)$ such that $(*)$ solves the inhomogeneous differential equation:

$y' = \alpha'(x) e^{-\cos x} + \cancel{\alpha(x) \sin x\, e^{-\cos x}}$

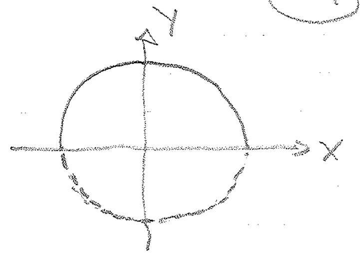
$= \cancel{\sin(x)\, \alpha(x) e^{-\cos x}} + 2x\, e^{-\cos x}$

$\Rightarrow \alpha'(x) \cancel{e^{-\cos x}} = 2x \cancel{e^{-\cos x}} \Rightarrow \alpha(x) = x^2 + C_1'$

$y = (x^2 + C_1')\, e^{-\cos x}$

e) Exact differential equations and integrating factors

Example 1:



$x^2 + y^2 = C_1'^2$ circle with radius C_1'

$y = \pm\sqrt{C_1'^2 - x^2}$

Implicit differentiation

$2x + 2y \frac{dy}{dx} = 0 \Rightarrow y' = -\frac{x}{y}$ differential equation

(with general solution $y = \pm\sqrt{C^2 - x^2}$)

$\Rightarrow$ generalisation?

Remark (on partial derivatives):

$F(x,y) = x^2 y + \cos(xy)$ (function of <u>two</u> variables x, y <u>both</u> independent)

$\frac{\partial F}{\partial x} = 2xy - y\sin(xy)$ (y fixed!)

$\frac{\partial F}{\partial y} = x^2 - x\sin(xy)$ (x fixed!)

Second derivatives

$\frac{\partial^2 F}{\partial y \partial x} = 2x - \sin(xy) - xy\cos(xy)$

$\frac{\partial^2 F}{\partial x \partial y} = 2x - \sin(xy) - xy\cos(xy)$

$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$

Chainrule: if $y = y(x)$ then $F(x, y(x))$ is a function of x. Derivative with respect to x ("implicit differentiation")

$$\frac{dF(x,y(x))}{dx} = \frac{d(x^2y)}{dx} + \frac{d\cos(xy)}{dx} = 2xy + x^2y' - (xy)'\sin(xy)$$

$$= 2xy + x^2y' - y\sin(xy) - xy'\sin(xy)$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y'$$

General case (cf Ex 1):

Let $F(x,y) = C_1$

Differentiation yields

$$0 = \frac{dF(x,y)}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

Call $\frac{\partial F}{\partial x} = P(x,y) \qquad \frac{\partial F}{\partial y} = Q(x,y)$

then $\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial Q}{\partial x}$

and $0 = P(x,y) + Q(x,y)\cdot y'$

Summary: The differential equation

$$P(x,y) + Q(x,y)y' = 0 \qquad \left(\text{i.e. } y' = -\frac{P(x,y)}{Q(x,y)}\right)$$

is called exact differential equation if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The implicit form of the solution is given by

$$F(x,y) = C_1$$

where F is determined by

$$\frac{\partial F}{\partial x} = P(x,y) \qquad \frac{\partial F}{\partial y} = Q(x,y)$$

Example 2: $y' = -\frac{x}{y} \Rightarrow \underbrace{x}_{P} + \underbrace{y}_{Q}\cdot y' = 0$

$P(x,y) = x,\ Q(x,y) = y \qquad \frac{\partial P}{\partial y} = 0 \quad \frac{\partial Q}{\partial x} = 0 \Rightarrow$ exact diff. eq. ✓

$F = ?$ $\frac{\partial F}{\partial x} = P(x,y) = x \Rightarrow F(x,y) = \int x\,dx = \frac{1}{2}x^2 + C_1(y)$!!!

$\frac{\partial F}{\partial y} = Q(x,y) = y \Rightarrow \frac{dC_1(y)}{dy} = y \Rightarrow C_1(y) = \frac{1}{2}y^2 + C_2$!!

$\Rightarrow F(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C_2$ ("first integral")

Solution

$F(x,y) = C_1 \Rightarrow x^2 + y^2 = \frac{C_1 - C_2}{2} = D$ (cf. Ex 1)

Example 3: $\sin(x+y) + \{2y + \sin(x+y)\}y' = 0$

$P(x,y) = \sin(x+y) \qquad \frac{\partial P}{\partial y} = \cos(x+y)$

$Q(x,y) = 2y + \sin(x+y) \qquad \frac{\partial Q}{\partial x} = \cos(x+y)$ ✓ exact.

$\frac{\partial F}{\partial x} = P(x,y) = \sin(x+y) \Rightarrow F(x,y) = \int \sin(x+y)\,dx$!!!

$= -\cos(x+y) + C'_1(y)$

$\frac{\partial F}{\partial y} = Q(x,y) \Rightarrow \sin(x+y) + \frac{dC_1(y)}{dy} = 2y + \sin(x+y)$

$\Rightarrow \frac{dC_1}{dy} = 2y \Rightarrow C_1(y) = y^2 + C'_2$