MTH4102 DIFFERENTIAL EQUATIONS

Module Website

http://www.maths.qmul.ac.uk/~jel/MTH4102/index.html

- contains useful information, notices, courseworks

Module Text

J. Polking, A. Boggess, D. Arnold:
Differential Equations (Pearson 2006)

Note: Exercise classes start in week 2

- Check website for class allocation (by surname)
- Courseworks contain in-class problems and homeworks (labelled A, B, C etc) to be handed in for marking

In this module we are interested in <u>differential equations</u> and their <u>solutions</u>

<u>What is a differential equation?</u>

A differential equation relates an unknown <u>function</u> $y(x)$ of a <u>variable</u> x to the <u>derivatives</u> of the function.

<u>e.g.</u>

$$\frac{dy}{dx} = y^2 - x^2$$

derivative
'rate of change'

some function of y and x

Problem is to determine the $y(x)$ that satisfies the differential equation. This is the <u>solution</u>.

Why consider? They describe our world
eg. physics, electronics, chemistry, solar system, economics.

- They have nice mathematical properties

Functions

Let $y = g(x)$

x: independent variable

y: dependent variable

$g(x)$: function

Definition – A function is a rule that relates variables from a domain ($x \in D$) with variables of a range ($y \in R$)

$$g : D \to R \quad \text{and} \quad x \mapsto y = g(x)$$

Example: Let $y = g(x) = -x^2 + 4$

$$g(-3) = -(-3)^2 + 4 = -5$$

$$g(a) = -a^2 + 4$$

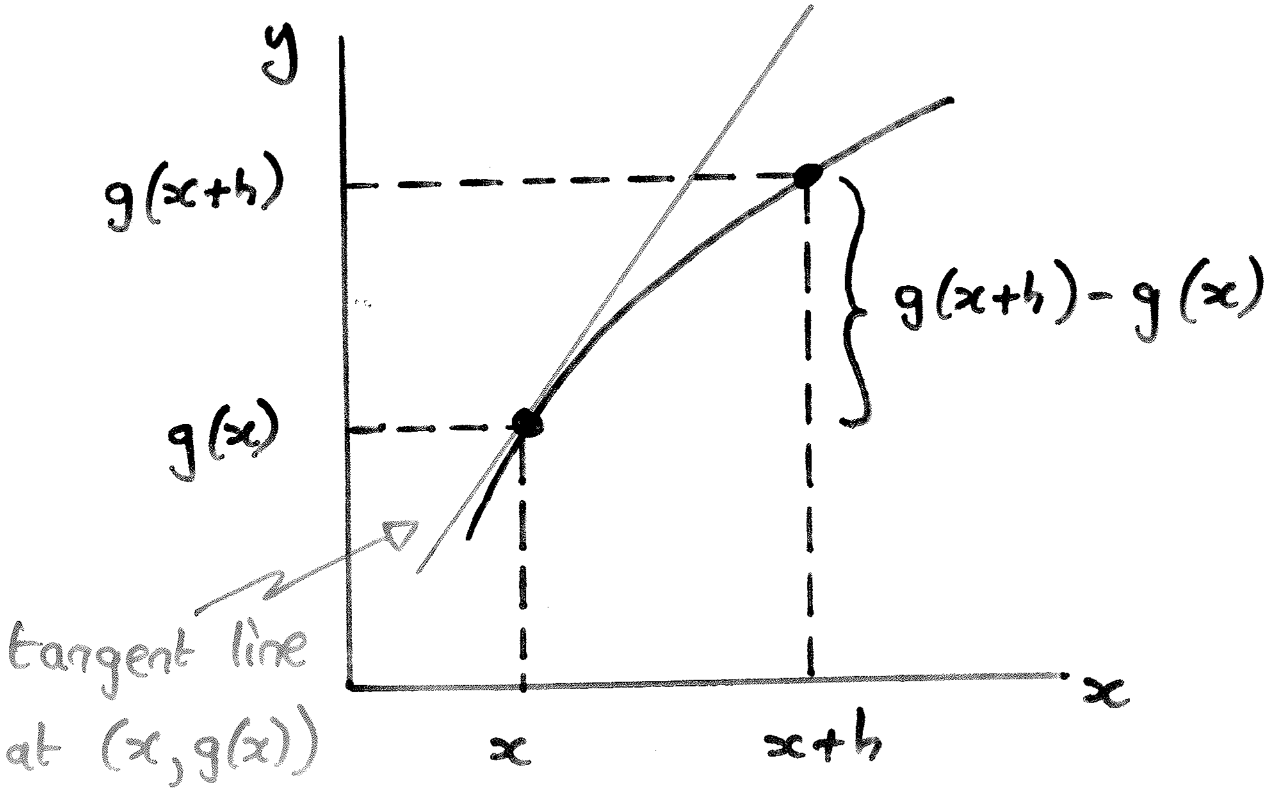
Example: $y = g(x) = x^3 - x$

$$g(1) = 1^3 - 1 = 0$$

Remark: Normally we use notation $y = g(x)$ and write $y(x)$ or y to denote function $g(x)$

Derivative : What is it?

1. Rate of change of function
2. Slope of tangent line to graph of a function



Consider : $$m = \frac{g(x+h) - g(x)}{h}$$

This is gradient of line through points $(x, g(x))$ and $(x+h, g(x+h))$

Define

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \frac{dg}{dx}$$

$$= \frac{dy}{dx} = y'$$

Remark : $g'(x)$ determines rate of change of function $g(x)$ at point x :

$$dg = g'(x)\,dx$$

Rules of Differentiation

Often write $g'(x)$ to denote $\frac{dg}{dx}$

Let $f(x)$ and $g(x)$ be functions of x

Product Rule : $(fg)' = f'g + fg'$

Quotient Rule : $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain Rule : If $h(x)$ is function of x and g is a function of $h(x)$ such that $g = g(h(x))$, then

$$g' = \frac{dg}{dx} = \frac{dg}{dh}\,\frac{dh}{dx}$$

Example : $y(x) = x^3 \sin(x^2)$

$$y'(x) = 3x^2 \sin(x^2) + 2x^4 \cos(x^2)$$

Indefinite Integral (Anti-derivative)

A function $G(x)$ is called anti-derivative of function $y = g(x)$ if $G'(x) = \frac{dG}{dx} = y(x)$

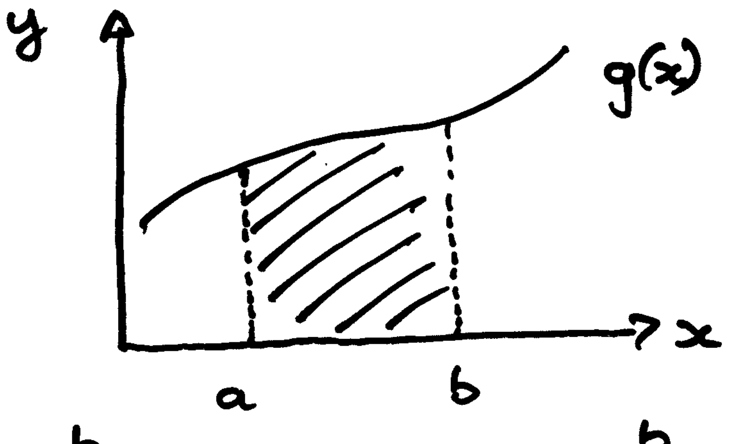
Example :

$$\left.\begin{aligned} G(x) &= \frac{x^3}{3} \\ G(x) &= \frac{x^3}{3} - 2 \end{aligned}\right\} \text{ both anti-derivatives of } g(x) = x^2$$

__In general__ : Anti-derivative given by

$$\mathcal{G}(x) = \int g(x)\,dx + \underbrace{C}_{\text{arbitrary integration constant. (IMPORTANT)}}$$

__Definite Integral__ : Area under graph of function



$$\text{Area} = \int_a^b g(x)\,dx = \mathcal{G}(x)\Big|_a^b = \mathcal{G}(b) - \mathcal{G}(a)$$

__Example__ : $y = g(x) = \sqrt{x}$

$$\mathcal{G}(x) = \frac{2}{3}x^{3/2} + C$$

$$\int_1^2 x^{1/2}\,dx = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

Integration by Parts

Let $u = u(x)$ and $v = v(x)$ be functions of x. Then

$$\int u\,dv = uv - \int v\,du$$

Example: Integrate $\int x e^{x}\,dx$

Let $u = x$ and $dv = e^{x}\,dx$

Then $\int x e^{x}\,dx = x e^{x} - e^{x} + C$

Example: Integrate $\int x \sin x\,dx$

Let $u = x$ and $dv = \sin x\,dx$

Then $\int x \sin x\,dx = -x\cos x + \sin x + C$

Initial Value Problem

A differential equation relates derivative $y' = \frac{dy}{dx}$ to dependent and independent variables. In general,

$$\frac{dy}{dx} = \underbrace{f(x,y)}_{\text{function of } x \text{ and } y}$$

Problem: How to find solution $y(x)$?

Example: $$y' = \frac{dy}{dx} = x^2 + 3$$

Try integration. Anti-derivative is

$$y(x) = \int \frac{dy}{dx}\, dx = \int (x^2+3)\, dx$$

$$= \frac{1}{3}x^3 + 3x + C$$

This called general solution to this differential equation.

Determine constant of integration, C, from a given <u>initial condition</u>

<u>Example</u>: Suppose $y(0) = \frac{1}{3}$

$$\Rightarrow \quad \frac{1}{3} = y(0) = 0 + C \quad \Rightarrow \quad C = \frac{1}{3}$$

Then $y = \frac{1}{3}x^3 + 3x + \frac{1}{3}$ is called a <u>particular solution</u> that satisfies the differential equation with particular initial condition.

<u>In general</u>, a differential equation

$\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$

is called <u>initial value problem</u>. The function $y = y(x)$ that satisfies the problem known as <u>solution</u> to initial value problem.

First-Order Differential Equations

The **order** of a differential equation is the order of the highest derivative that appears.

Example :

$$\left.\begin{aligned} \frac{dy}{dx} &= y^2+x \\ \left(\frac{dy}{dx}\right)^2 &= y^2-x^2 \end{aligned}\right\} \text{first-order } \left(\frac{dy}{dx} \text{ only}\right)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} - x \quad : \text{second-order}$$

A first order differential equation is in normal form if it can be written as

$$y' = \frac{dy}{dx} = f(x,y)$$

Example : $y' - y^2 = x$ is normal, $f = y^2 + x$

$(y')^2 = yx$ not normal form

Methods to find Solutions

A solution of first-order differential equation $y' = f(x,y)$ is the function $y = y(x)$ such that $y' = f(x, y(x))$ for all x where $y(x)$ is defined.

Solution by Substitution

Idea : guess form of solution (possibly upto arbitrary constant to be determined by substitution).

Example : Find solution to

$$y' = -\frac{y}{1+x}$$

Try $y(x) = 1+x$ (not solution)

Try $y(x) = \frac{1}{1+x}$ (is a solution !)

Example : determine value of the constant, a, such that $y(x) = e^{ax}$ is a solution to

$$y' = -2y$$

Solution is $y = e^{-2x}$

Solution by Direct Integration

Works for differential equation of form

$$y' = \frac{dy}{dx} = f(x)$$

where function f depends only on independent variable, x.

Idea : Find anti-derivative. In practise treat $\frac{dy}{dx}$ as a fraction and integrate both sides.

In general, $\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)\,dx$

$$\Rightarrow \int dy = \int f(x)\,dx$$

$$\Rightarrow \quad y(x) = \int f(x)\,dx + C$$

Remember to include arbitrary integration constant !

Example : Determine solution of initial value problem

$$y' = \frac{1}{(x-2)(x-3)} \quad , \; y(0) = -1$$

Partial Fractions : $\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

$$A(x-3) + B(x-2) = 1 \quad \Rightarrow \quad A = -1 \; , \; B = 1$$

$$y = \int \frac{dx}{(x-2)(x-3)} = \int \frac{dx}{x-3} - \int \frac{dx}{x-2}$$

$$y(x) = \ln|x-3| - \ln|x-2| + C$$

(general solution)

Particular Solution

$$y(0) = -1 = \ln 3 - \ln 2 + C$$

$$C = \ln 2 - \ln 3 - 1$$

$$y(x) = \ln\left|\frac{x-3}{x-2}\right| + \ln\left(\frac{2}{3}\right) - 1$$