What if $y' = f(x,y)$?

Example : find solution of

$$y' = \frac{dy}{dx} = x^2 - y^2$$

By integration : $dy = (x^2 - y^2)dx$

$$\Rightarrow \int dy = y = \int (x^2 - y^2)dx$$

$$= \frac{1}{3}x^3 - \int (y(x))^2 dx$$

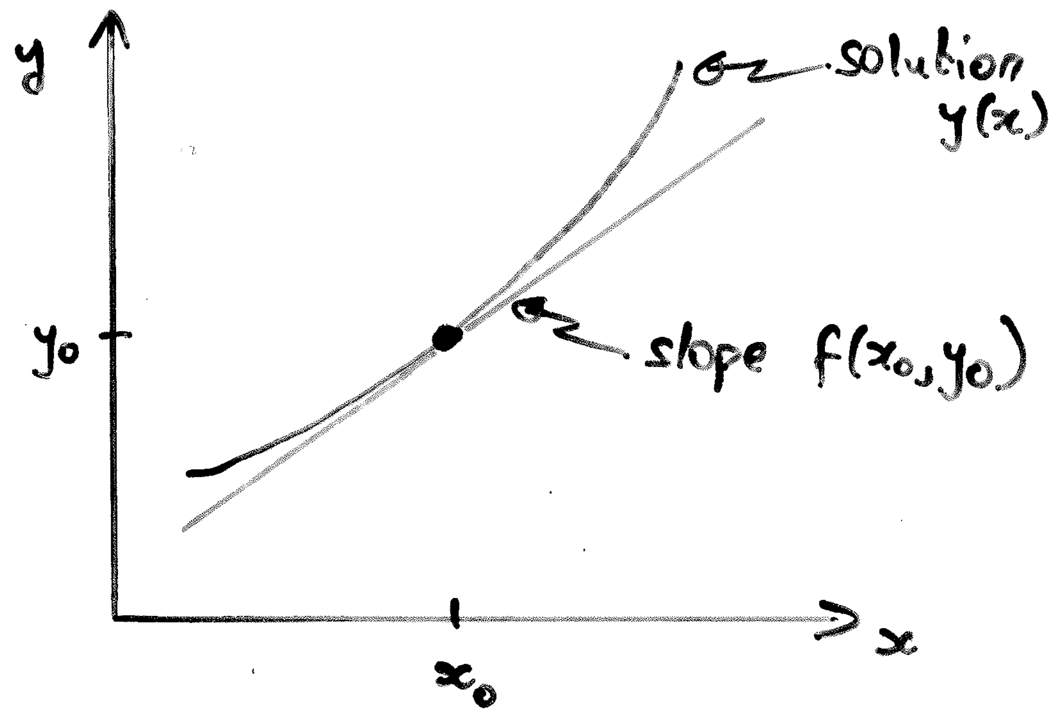
Problem : $y(x)$ unknown

Direction Field

Uses geometric interpretation. First-order differential equation

$$y' = f(x, y(x))$$

says that solution $y(x)$ has slope (gradient) $f(x,y)$ at point (x,y)



$f(x_0, y_0)$ determines slope of solution at (x_0, y_0).

Idea: draw a slanted line segment with slope $f(x,y)$ at each point (x,y). Called direction field. A solution of the differential equation is a curve which at each point is tangential to the direction field.

Positive Slope draw 

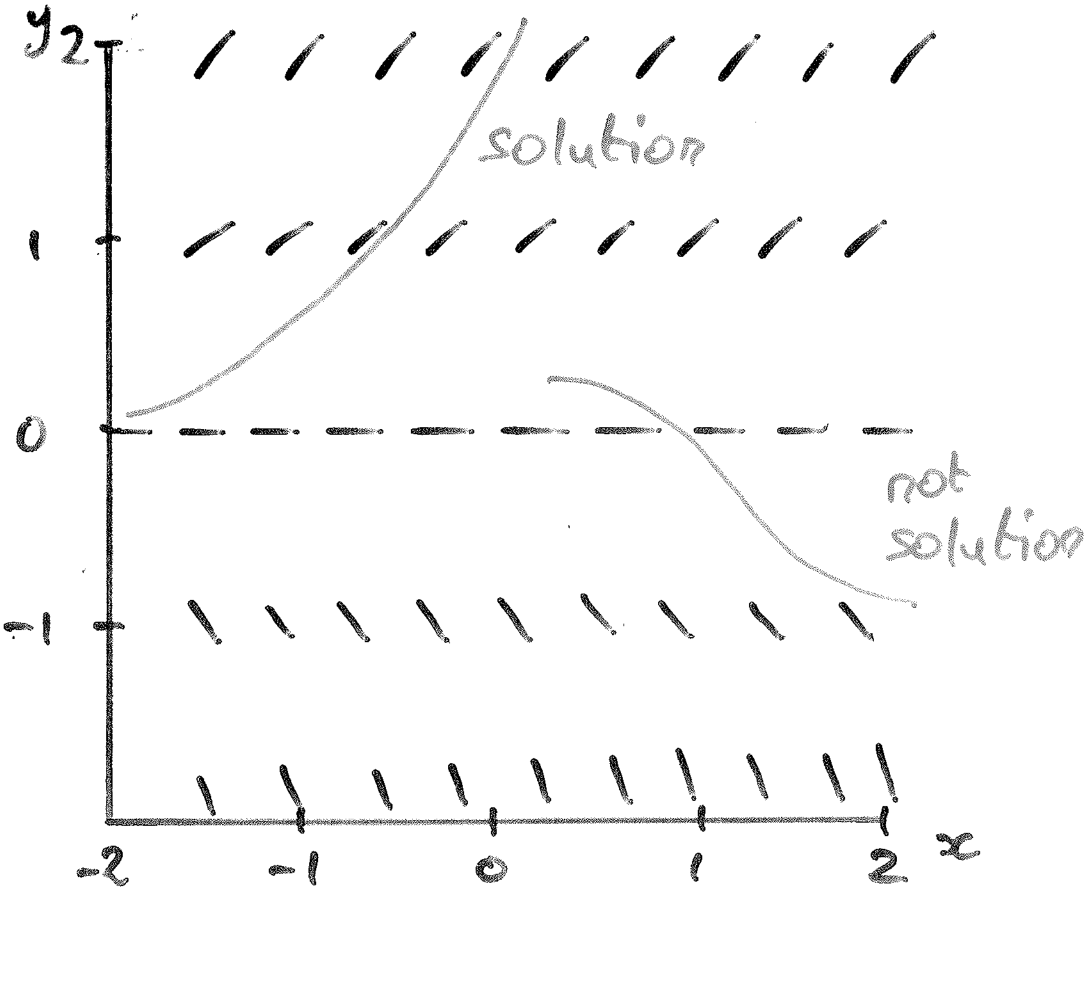
Negative Slope draw 

Zero Slope draw

<u>Example</u> : Sketch the direction field in the domain $x \in [-2,2]$, $y \in [-2,2]$ for the differential equation

$$y' = \frac{dy}{dx} = y$$

<u>Method</u> : Identify points in (x,y) plane where $y' = 0$, $y' > 0$ and $y' < 0$.



Example : Sketch direction field of differential equation

$$y' = f(x,y) = y(1-y)$$

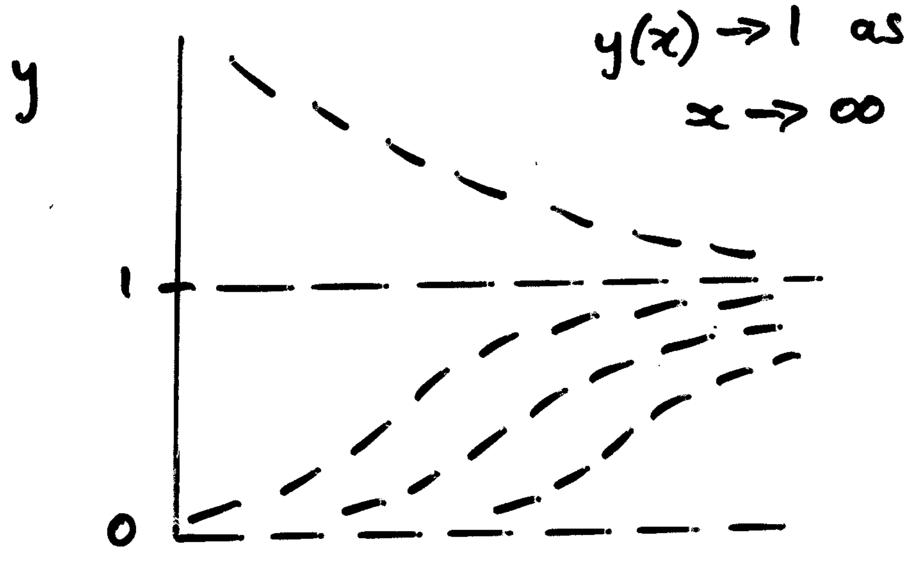
in the domain $x > 0,\ y > 0$

$y(0) = 0 \Rightarrow y(x) = 0$

$y(0) = 1 \Rightarrow y(x) = 1$

$0 < y(x) < 1 \Rightarrow y' > 0 \Rightarrow$ all direction fields slanted / but get horizontal as $y \to 1$

$y(x) > 1 \Rightarrow y' < 0$



Separation of Variables

Works for first-order differential equation of form

$$y' = f(x,y) = g(x)h(y)$$

where right-hand side is a product of function depending only on x and a function depending only on y

Idea : Separate variables and treat dy/dx as a 'fraction.' Put all y-dependence on one side and all x-dependence on other side. Then integrate

Example : Solve initial value problem

$$\frac{dy}{dx} = y \qquad y(0) = 3$$

$$\Rightarrow \frac{dy}{y} = dx \Rightarrow \int \frac{dy}{y} = \int dx$$

$$\Rightarrow \ln|y| = x + C$$ (general solution)

Solve for y : exponentiate

$$e^{\ln|y|} = |y| = e^{x+c} = e^c e^x$$

$$\Rightarrow \quad y = +e^c e^x \quad \text{or} \quad y = -e^c e^x$$

$$\Rightarrow \quad y = De^x \qquad D = \text{arbitrary constant}$$

Determine arbitrary constant from initial condition

$$y(0) = 3 \Rightarrow 3 = De^0 \Rightarrow D = 3$$

$$\Rightarrow \quad y = 3e^x$$

This is solution to initial value problem.

<u>Remark</u>: An <u>explicit</u> solution is one that can be expressed as a function of the independent variable, $y = y(x)$. An <u>implicit</u> solution is one where such a formula does not exist. e.g.

$$ye^y = x$$

Example: Solve Initial Value problem

$$\frac{dy}{dx} = xy^2 \quad , \quad y(0) = -1$$

Example:

Find solution to first-order diff. equation

$$\frac{dy}{dx} = \frac{2xy}{1+y}$$

subject to initial condition

$y(0) = 2$

Why does separation of variables work?

In general, if $y' = g(x)h(y)$ then

$$\frac{1}{h(y)} y'(x) = g(x) \Rightarrow \frac{1}{h(y)} y'(x)\,dx = g(x)\,dx$$

$$\int \frac{1}{h(y)} y'(x)\,dx = \int g(x)\,dx$$

If $y = y(x)$ denotes solution, $dy = \frac{dy}{dx}dx = y'dx$

yields

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Remark: In principle works for arbitrary functions $g(x)$, $h(y)$ but integrals may be difficult to solve. May also be difficult or impossible to obtain explicit solution $y = y(x)$

Change of Variables

Works for diff. equations of form

$$\frac{dy}{dx} = y' = f\left(\frac{y}{x}\right)$$

Idea: define new dependent variable to rewrite equation in separable form

Define function $z(x)$:

$$y(x) = x\,z(x)$$

$$\Rightarrow \quad \frac{dy}{dx} = z + x\frac{dz}{dx}$$

Substitute:

$$z + x\frac{dz}{dx} = f(z)$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{f(z) - z}{x} = \frac{g(z)}{x}$$

Separable!

Example

Find general solution $y = y(x)$ to first-order diff. equation

$$y' = \frac{dy}{dx} = \frac{y}{x} + 1$$

$$y = xz(x) \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\Rightarrow \quad z + x\frac{dz}{dx} = z + 1$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{1}{x}$$

$$\int dz = z(x) = \int \frac{dx}{x} = \ln|x| + C$$

$$\Rightarrow \quad \frac{y(x)}{x} = \ln|x| + C$$

$$y(x) = x(\ln|x| + C)$$

First-Order Linear Differential Equations

Equation of the form

$$y' = a(x)y$$

called <u>linear</u>, <u>homogeneous</u> diff. equation.

Equation of the form

$$y' = a(x)y + b(x)$$

called <u>linear</u>, <u>inhomogeneous</u> diff. equation

<u>Remark</u>: right-hand side must depend only linearly on $y(x)$. Otherwise equation is <u>non-linear</u>

<u>Example</u>: $y' = \sin(x)y$, homogeneous

$y' = e^{x}y + x$, inhomogeneous

$y' = x\sin(y)$, non-linear

Solution to Homogeneous Equations

Can be solved explicitly

Idea : use separation of variables.

$$\frac{dy}{dx} = a(x)y \Rightarrow \frac{dy}{y} = a(x)\,dx$$

$$\Rightarrow \int \frac{dy}{y} = \int a(x)\,dx$$

Define : $A(x) = \int a(x)\,dx$

$$\Rightarrow \ln|y| = A(x) + C$$

$$\Rightarrow |y| = e^{C} e^{A(x)}$$

Define : $D = e^{C}$

$$y = De^{A(x)} = De^{\int a(x)\,dx}$$

This is general solution for arbitrary function $a(x)$

Example: Solve initial value problem

$$\frac{dy}{dx} = \cos(x)y \quad , \quad y\left(\frac{\pi}{2}\right) = 2$$

$$\frac{dy}{y} = \cos(x)\,dx \Rightarrow \int \frac{dy}{y} = \int \cos(x)\,dx$$

$$\Rightarrow \ln|y| = \sin(x) + C$$

$$\Rightarrow |y| = e^{C} e^{\sin(x)}$$

$$\Rightarrow y(x) = De^{\sin(x)}.$$

$$y\left(\frac{\pi}{2}\right) = 2 \Rightarrow 2 = De^{\sin\left(\frac{\pi}{2}\right)} = De^{1}$$

$$\Rightarrow D = 2e^{-1}$$

Particular solution is

$$y(x) = 2e^{-1}e^{\sin(x)}$$
$$= 2e^{\sin(x)-1}$$