Question 2.3
Let W denote the boiling point of water measured in $^\circ C$ and Y in $^\circ F$. Then we can write

$$Y_i = \frac{9}{5} W_i + 32.$$

Also, denote by $\widehat{\beta}_{1F}$ the estimator of β_1 obtained for observations in Fahrenheit and by $\widehat{\beta}_{1C}$ the estimator obtained for observations in Celsius. Then

$$\begin{aligned} \widehat{\beta}_{1F} &= \sum_{i=1}^{n} \frac{x_i - \overline{x}}{S_{xx}} Y_i = \sum_{i=1}^{n} c_i Y_i \\ &= \sum_{i=1}^{n} c_i \left(\frac{9}{5} W_i + 32 \right) = \frac{9}{5} \sum_{i=1}^{n} c_i W_i + 32 \sum_{i=1}^{n} c_i \\ &= \frac{9}{5} \sum_{i=1}^{n} c_i W_i = \frac{9}{5} \widehat{\beta}_{1C} \end{aligned}$$

by $\sum_{i=1}^{n} c_i = 0$, where $c_i = \frac{x_i - \overline{x}}{S_{xx}}$. The estimators are proportional, that is

$$\widehat{\beta}_{1F} = \frac{9}{5} \widehat{\beta}_{1C}.$$

Question 3. We have

$$\begin{aligned} e_i &= Y_i - \widehat{Y}_i = Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i = Y_i - \overline{Y} - \widehat{\beta}_1 (x_i - \overline{x}) \\ &= Y_i - \overline{Y} - \sum_{j=1}^{n} c_j Y_j (x_i - \overline{x}) = Y_i - \frac{1}{n} \sum_{j=1}^{n} Y_j - \sum_{j=1}^{n} c_j Y_j (x_i - \overline{x}) \\ &= Y_i - \sum_{j=1}^{n} \left[\frac{1}{n} + (x_i - \overline{x}) c_j \right] Y_j \\ &= \left\{ 1 - \left[\frac{1}{n} + (x_i - \overline{x}) c_i \right] \right\} Y_i - \sum_{j \neq i} \left[\frac{1}{n} + (x_i - \overline{x}) c_j \right] Y_j \end{aligned}$$

This is a linear combination of independent random variables Y_j and so the variance of the combination is the combination of variances of Y_j with squared coefficients. Furthermore, $\text{var}(Y_j) = \sigma^2$, $j = 1, \ldots, n$. Hence,

$$\begin{aligned} \text{var}(e_i) &= \sigma^2 \left\{ 1 - \left[\frac{1}{n} + (x_i - \overline{x}) c_i \right] \right\}^2 + \sigma^2 \sum_{j \neq i} \left[\frac{1}{n} + (x_i - \overline{x}) c_j \right]^2 \\ &= \sigma^2 \left\{ 1 - 2 \left[\frac{1}{n} + (x_i - \overline{x}) c_i \right] + \left[\frac{1}{n} + (x_i - \overline{x}) c_i \right]^2 + \sum_{j \neq i} \left[\frac{1}{n} + (x_i - \overline{x}) c_j \right]^2 \right\} \\ &= \sigma^2 \left\{ 1 - 2 \left[\frac{1}{n} + (x_i - \overline{x}) c_i \right] + \sum_{j=1}^{n} \left[\frac{1}{n} + (x_i - \overline{x}) c_j \right]^2 \right\} \\ &= \ldots = \sigma^2 \left[1 - \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}} \right) \right] \end{aligned}$$

by $\sum_{j=1}^{n} c_j = 0$ and $\sum_{j=1}^{n} c_j^2 = \frac{1}{S_{xx}}$.