MTH5103 COMPLEX VARIABLES
MID-TERM TEST

Date: 23 February 2012 *Time:* 10.10–10.50

Complete the following information:

Name	
Student Number (9 digit code)	

The test has FOUR questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish marked. Calculators are NOT allowed.

Question	Marks
A	
B	
C	
D	
Total Marks	

Question A. [*20 marks*]
Let $z_1 = -1 + 5i$ and $z_2 = 2e^{3i\pi/4}$. Compute:

(a) $\mathrm{Im}(z_2)$, [*4 marks*]

(b) $|z_1|$, [*4 marks*]

(c) $|z_1 + z_2|$, [*6 marks*]

(d) z_1^{-1} . [*6 marks*]

There is no need to evaluate radicals. Present your result in the standard form $z = x + iy$ in (d).

Answer A.

$$Z_2 = 2\cos\frac{3\pi}{4} + 2i\sin\frac{3\pi}{4} = \frac{-2+2i}{\sqrt{2}} \quad (3) \Rightarrow \mathrm{Im}\, Z_2 = \sqrt{2} \quad (1)$$

$$|Z_1| = \sqrt{1^2+5^2} = \sqrt{26} \quad (4)$$

$$|Z_1 + Z_2| = |-1+5i-\sqrt{2}+\sqrt{2}i|$$

$$= \sqrt{(-1-\sqrt{2})^2 + (5+\sqrt{2})^2} \quad (3)$$

$$= \sqrt{1+2+2\sqrt{2}+25+2+10\sqrt{2}} = \sqrt{30+12\sqrt{2}} \quad (3)$$

$$Z_1^{-1} = \frac{1}{-1+5i} = \frac{-1-5i}{(-1)^2+5^2} = -\left(\frac{1+5i}{26}\right) \quad (6)$$

Question B. [*35 marks*]

(a) Find all complex solutions of $z^4 = -16i$, giving the results in polar form and indicating their location in the Argand diagram. [*15 marks*]

(b) Derive the curve corresponding to

$$|z+2i| + |z-2i| = 6$$

[*20 marks*]

and describe this geometrically.

Answer B.

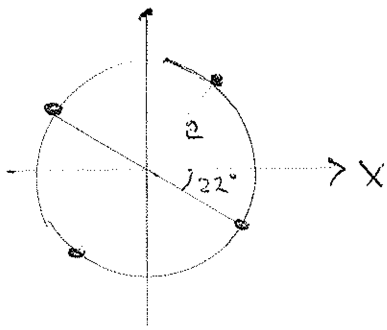
(3)

$$z^4 = -16i = 2^4 e^{-\frac{i\pi}{2} + 2\pi k i} \quad (k = 0, 1, 2, 3)$$

$$\Rightarrow z = 2e^{-\frac{i\pi}{8} + \frac{\pi k i}{2}}$$

$$= 2e^{-i\pi/8}, 2e^{i3\pi/8}, 2e^{i7\pi/8}, 2e^{i11\pi/8} \quad (7)$$

i.e. 4 roots lie on circle of radius 2 with separation of 90° (5)



(3)

$$|x+iy+2i| + |x+iy-2i| = 6$$

$$\Rightarrow \sqrt{x^2+(y+2)^2} = 6 - \sqrt{x^2+(y-2)^2}$$

Squaring: $\cancel{x^2} + \cancel{y^2} + 4y + \cancel{4} = 36 + \cancel{x^2} + \cancel{y^2} - 4y + \cancel{4} - 12\sqrt{x^2+(y-2)^2}$

$$\Rightarrow \overset{9}{\cancel{36}} - \overset{2}{\cancel{8}}y = \overset{3}{\cancel{12}}\sqrt{x^2+(y-2)^2}$$

Squaring: $81 - \cancel{36}y + 4y^2 = 9x^2 + 9y^2 - \cancel{36}y + 36$ (15)

$$\Rightarrow 9x^2 + 5y^2 = 45 \Rightarrow \left(\frac{x}{\sqrt{5}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

This is an ellipse. (2)

Question C. [*25 marks*]

(a) Evaluate the limit

$$\lim_{z \to i/2} \frac{(2z-3)(4z+i)}{(iz-1)^2}.$$

[*10 marks*]

(b) Starting from the definition of the derivative of a complex function as a limit, find the derivative of

$$f(z) = 1/z^2$$

for $z \in \mathbb{C}$ and state where $f'(z)$ is holomorphic. [*15 marks*]

Answer C.

$$\lim_{z \to i/2} \frac{(2z-3)(4z+i)}{(iz-1)^2} = \frac{\lim(2z-3) \times \lim(4z+i)}{[\lim(iz-1)]^2} \quad (2)$$

$$= \frac{(i-3)\times(2i+i)}{\left(-\frac{1}{2}-1\right)^2} \quad (3) \quad = \frac{3i(i-3)}{\left(\frac{9}{4}\right)} = -\frac{4}{3} - 4i \quad (4)$$

$$f'(z) = \lim_{\Delta z \to 0}\left\{\frac{f(z+\Delta z) - f(z)}{\Delta z}\right\} \quad (2)$$

$$= \lim_{\Delta z \to 0}\left\{\frac{(z+\Delta z)^{-2} - z^{-2}}{\Delta z}\right\} = \lim_{\Delta z \to 0}\left\{\frac{z^{-2}\left(1+\frac{\Delta z}{z}\right)^{-2} - z^{-2}}{\Delta z}\right\}$$

$$= \lim_{\Delta z \to 0}\left\{\frac{\cancel{z^{-2}} - \frac{2\Delta z}{z^3} + \frac{3(\Delta z)^2}{z^4} - \cdots - \cancel{z^{-2}}}{\Delta z}\right\} = -\frac{2}{z^3}$$

(9)

Holomorphic except at $z = 0$ (3)

Question D. [20 *marks*]

(a) Let $f(z) = u + iv$, where $z = x + iy$. Write down the *Cauchy-Riemann* equations relating u and v. What other conditions are required for $f(z)$ to be differentiable at z? [*10 marks*]

(b) Use the *Cauchy-Riemann* equations to find the set of points at which

$$f(z) = x^2 + iy^3$$

is differentiable. [*10 marks*]

Answer D.

CR eqns: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (4)

Also require $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ to exist and be continuous in small disc around Z (6)

$f = x^2 + iy^3 \Rightarrow u = x^2, \; v = y^3$ (2)

Then $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 3y^2$ (4)

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 3y^2$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ satisfied everywhere. (4)