Question A. [*20 marks*]
Let $z_1 = -2+3i$ and $z_2 = 4e^{i\pi/3}$. Compute:

(a) $\mathrm{Im}(z_2)$, [*4 marks*]

(b) $|z_1|$, [*4 marks*]

(c) $|z_1 - z_2|$, [*6 marks*]

(d) z_1^{-1} . [*6 marks*]

There is no need to evaluate radicals. Present your result in the standard form $z = x + iy$ in (d).

Answer A.

$$Z_2 = 4\cos\frac{\pi}{3} + 4i\sin\frac{\pi}{3} = 2 + 2\sqrt{3}\,i \Rightarrow \mathrm{Im}\,Z_2 = 2\sqrt{3}$$ 2

$$|Z_1| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$ 2

$$|Z_1 - Z_2| = |-2+3i-2-2\sqrt{3}i|$$

$$= \sqrt{(-4)^2 + (3-2\sqrt{3})^2}$$

$$= \sqrt{16+9-12\sqrt{3}+12} = \sqrt{37-12\sqrt{3}}$$ 3

$$Z_1^{-1} = \frac{1}{-2+3i} = \frac{-2-3i}{(-2)^2+3^2} = -\left(\frac{2+3i}{13}\right)$$ 3

Question B. [*32 marks*]

(a) Find all complex solutions of $z^3 = 8i$, giving the results in polar form and indicating their location in the Argand diagram. [*16 marks*]

(b) Derive the curve corresponding to

$$\left|\frac{z-1}{z+1}\right| = 2$$

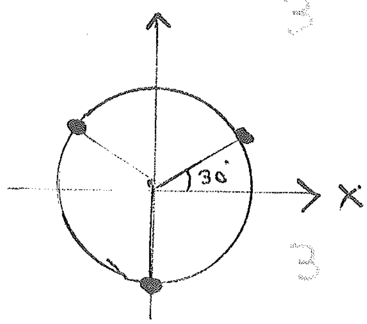
and describe this geometrically. [*16 marks*]

Answer B.

$$z^3 = 8i = 2^3 e^{\frac{i\pi}{2} + 2\pi k i} \quad (k = 0, 1, 2)$$ 2

$$\Rightarrow z = 2e^{\frac{i\pi}{6} + \frac{2\pi k i}{3}}$$ 3

$$= 2e^{i\pi/6},\ 2e^{5i\pi/6},\ 2e^{3i\pi/2}$$



i.e. 3 roots lie on circle of radius 2 with separation 120° 3

$$\left|\frac{z-1}{z+1}\right| = \left|\frac{(x-1)+iy}{(x+1)+iy}\right| = \sqrt{\frac{(x-1)^2+y^2}{(x+1)^2+y^2}} = 2$$ 2

$$\Rightarrow (x-1)^2 + y^2 = 4(x+1)^2 + 4y^2$$

$$\Rightarrow 3x^2 + 3y^2 + 10x + 3 = 0$$

$$\Rightarrow x^2 + \frac{10x}{3} + \frac{25}{9} + y^2 = \frac{25}{9} - 1 = \frac{16}{9}$$ 4

$$\Rightarrow \left(x + \frac{5}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2$$

This is circle with centre $\left(-\frac{5}{3}, 0\right)$ and radius $\frac{4}{3}$

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Question C. [*22 marks*]

(a) Evaluate the limit

$$\lim_{z\to i}\frac{(2z+7)(z+i)}{(z-2i)^3}.$$

[*10 marks*]

(b) Under the transformation

$$z\mapsto w=z^2+2z-5,$$

where $z=x+iy$ and $w=u+iv$, determine which points in the z-plane are mapped to $v=2$ in the w-plane and describe these geometrically.

[*12 marks*]

Answer C.

$$\lim_{z\to i}\left\{\frac{(2z+7)(z+i)}{(z-2i)^3}\right\}=\left\{\frac{\lim(2z+7)\times\lim(z+i)}{[\lim(z-2i)]^3}\right\}$$ 2

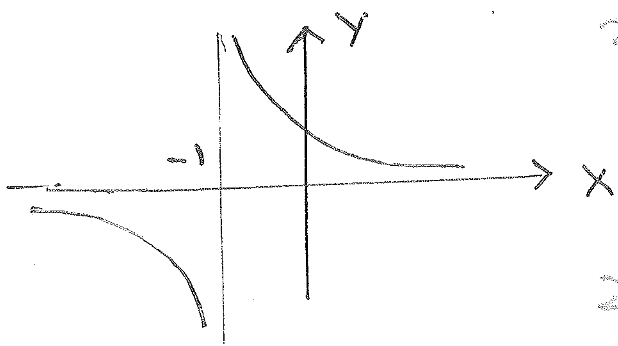
$$=\frac{(2i+7)\times(2i)}{(-i)^3}=2(2i+7)=14+4i$$ 3

$$W=z^2+2z-5=(x+iy)^2+2(x+iy)-5$$ 2

$$\Rightarrow u=x^2-y^2+2x-5,\quad v=2xy+2y$$ 2

$$v=2\Rightarrow y=\frac{1}{1+x}$$ 2

(i.e. rectangular hyperbola)



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Question D. [*26 marks*]

(a) Let $f(z) = u + iv$, where $z = x + iy$. Write down the *Cauchy-Riemann* equations relating u and v. What other conditions are required for $f(z)$ to be differentiable at z? [*8 marks*]

(b) Use the *Cauchy-Riemann* equations to find the values of z for which each of the following functions is differentiable:

$$f(z) = 3xy - ix^3, \quad f(z) = |z|^2$$

[*18 marks*]

Answer D.

CR eqns: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ 2

Also require $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ to exist 2

and be contin' at z $\boxed{\text{in small disc around } z}$.

$f = 3xy - ix^3 \Rightarrow u = 3xy, \; v = -x^3$ 1

$\dfrac{\partial u}{\partial x} = 3y = \dfrac{\partial v}{\partial y} = 0, \quad \dfrac{\partial u}{\partial y} = 3x = -\dfrac{\partial v}{\partial x} = 3x^2$ 2

Differentiable at $(0,0)$ and $(1,0)$ 2

$f = |z|^2 = x^2 + y^2 \Rightarrow u = x^2 + y^2, \; v = 0$ 1

$\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y} = 0, \quad \dfrac{\partial u}{\partial y} = 2y = -\dfrac{\partial v}{\partial x} = 0$ 2

Differentiable at $(0,0)$. 1